A Unified Approach to Boundary Value Problems

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ATHANASSIOS S. FOKAS
University of Cambridge
Cambridge, United Kingdom

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To I. M. Gel'fand, J. B. Keller, and P. D. Lax
for their lessons in precision, clarity, and
the search for unification

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Preface

The most well-known methods for the exact analysis of boundary value problems for linear PDEs are the methods of (a) classical transforms, (b) images, and (c) Green's function representations. In spite of their tremendous range of applications, these methods have several limitations, for example: (A) For *second order* PDEs and separable boundary conditions, the Sturm–Liouville theory establishes the existence and also provides an algorithmic construction of an appropriate transform, but for higher order PDEs, or for nonseparable boundary conditions, there do *not* exist appropriate transforms. For example, there does not exist an appropriate x-transform for the Dirichlet problem on the half-line for an evolution PDE involving a third order derivative. (B) The method of images is restricted to those particular problems that admit certain symmetries. (C) The linear integral equations arising in the implementation of the method of Green's function representations are difficult to solve in closed form.

In addition to these obvious limitations, there exist additional, subtle, disadvantages. For example, although the Dirichlet problem of the heat equation on the half-line can be solved by the sine transform in x, the associated solution representation is *not* uniformly convergent at $x = 0$. Hence, it is not straightforward to verify that the solution satisfies the prescribed Dirichlet boundary condition, and furthermore, this representation does not provide an effective algorithm for the numerical evaluation of the solution. A similar difficulty exists for the sine-series representation of the Dirichlet problem in a finite interval. Furthermore, for the finite interval with two different Robin boundary conditions, the solution is expressed through a series which involves eigenvalues satisfying a transcendental equation.

The situation is even less satisfactory for boundary value problems with conditions of "changing type," for example Dirichlet in part of a boundary and Neumann in the remaining part. For such problems, since there does *not* exist an appropriate transform, one uses *some* transform such as the Fourier transform, and then one tries to formulate a so-called Wiener–Hopf problem.

A new method for analyzing initial-boundary value problems for *integrable nonlinear* evolution PDEs was introduced by the author in [1]. It was later realized that this method also yields novel integral representations for *linear* evolution PDEs. For example, it yields novel integral representations even for the classical problem of the heat equation on the half-line. The first implementation of the new method to linear PDEs was *not* presented in the simplest possible form. The first goal of this book is to provide a simple and self-contained presentation for the case of linear PDEs, with particular emphasis on the following four points:

1. Novel integral representations for the solution of initial-boundary value problems for evolution PDEs containing x-derivatives of arbitrary order and which are formulated either on the half-line or the finite interval are presented in Part I. For the case that the PDE involves third order derivatives, the only alternative to the method presented here is the use of the Laplace transform in t. The best way for the interested reader to appreciate the advantage of the new method is to attempt to solve, via the Laplace transform, an initial-boundary value problem on the half-line for a PDE in $u(x,t)$ involving u_t, u_x, and u_{xxx}. For evolution PDEs, the main advantage of the new method is that it constructs integral representations which (a) are uniformly convergent; and (b) involve integrals in the complex k-plane, which via contour deformation can be mapped to integrals containing integrands which *decay exponentially*. This construction, in addition to providing effective asymptotic results, also yields a novel numerical technique which appears to have several advantages over the standard numerical methods.

2. Novel integral representations for the solution of the Laplace, the Helmholtz, and the modified Helmholtz equations formulated in the interior of a convex polygon are presented in Part III. These representations provide the basis for the development of certain analytical and numerical techniques for solving these PDEs. The example of the Dirichlet problem for the modified Helmholtz equation in the interior of an equilateral triangle is discussed in detail; the solution is expressed in terms of an integral in the complex k-plane which involves an integrand which *decays exponentially*.

3. It is emphasized throughout this book that the new approach provides a unification as well as a significant extension of the classical transforms, of the method of images, of the Green's function representations, and of the Wiener–Hopf technique. Regarding the latter technique we note that through a series of ingenious steps, it finally gives rise to the formulation of a Wiener–Hopf factorization problem, which is actually equivalent to a Riemann–Hilbert (RH) problem. It is shown in Part IV that such RH problems can be *immediately* obtained using the *global relation*. This relation, which is an algebraic equation coupling certain transforms of all boundary values, plays a crucial role in the new method.

4. An interesting byproduct of the new approach is the emergence of an effective method for inverting certain integrals. This method, which is based on the formulation of either an RH or a d-bar problem, provides an alternative and simpler approach for deriving classical transforms (such as the Fourier, the Mellin, and the Kontorovich–Lebedev transforms). Furthermore, it has led to the inversion of the so-called attenuated Radon transform; this transform provides the mathematical basis of an imaging technique of major medical importance called single photon emission computerized tomography (SPECT); see Part II.

The second goal of this book is to show that for *integrable nonlinear* evolution PDEs, the new method yields novel integral representations formulated in the complex k-plane. These integrals, in addition to the exponentials which appear in the integrals of the linearized version of these nonlinear PDEs, also contain the entries of a 2×2 matrix-valued function $M(x,t,k)$, which is the solution of a matrix RH problem. The main advantage of this formulation is that the associated RH problem involves a jump matrix with *explicit exponential* (x,t) *dependence*, and thus it is possible to obtain effective asymptotic results using the Deift–Zhou (for the long-time asymptotics) and the Deift–Zhou–Venakides (for the zero-dispersion limit) techniques for the asymptotic analysis of these RH problems. The analysis of initial-value problems on the half-line for the nonlinear Schrödinger, the

Korteweg–de Vries, the modified Korteweg–de Vries, and the sine-Gordon equations, as well as the crucial role played by the associated global relations, are discussed in Part V.

The main results contained in this book are summarized in the introduction, which contains five sections, each of which summarizes the results obtained in the corresponding part of the book, i.e., section I.1 corresponds to Part I, etc. The introduction is rather long, but perhaps it provides an opportunity for the interested reader to assimilate quickly the essential results of the book, thus avoiding many computational details.

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Athanassios S. Fokas

Introduction

Historical Remarks

The main elements of the method presented in this book were announced in 1997 (see [1]) after 15 years of efforts attempting to solve the following initial-boundary value problem suggested by the late Julian Cole to Mark Ablowitz and to the author during his visit to Clarkson University in 1982:

$$u_t + u_x + u_{xxx} + uu_x = 0, \qquad 0 < x < \infty, \qquad t > 0, \tag{1}$$

$$u(x,0) = u_0(x), \qquad 0 < x < \infty,$$

$$u(0,t) = 0, \qquad t > 0,$$

where $u(x,t)$ has sufficient decay for all t as $x \to \infty$ and $u_0(x)$ is a given smooth function with sufficient decay as $x \to \infty$ satisfying $u_0(0) = 0$. Equation (1) is the celebrated Korteweg–de Vries (KdV) equation which in the context of water waves describes irrotational, small amplitude, long waves; the first two terms of the left-hand side (LHS) of this equation describe the $O(1)$ contribution of waves traveling to the right; thus the term u_x *cannot* be neglected.

It was known in 1982 that the initial-value problem of the KdV equation can be solved by the so-called *inverse scattering transform method*, which was understood to have conceptual similarities with the Fourier transform method. Thus, we first tried to solve the linear version of the above initial-boundary value problem using an appropriate x-transform. The linear version of KdV satisfies

$$q_t + q_x + q_{xxx} = 0, \quad 0 < x < \infty, \quad t > 0, \tag{2a}$$

$$q(x,0) = q_0(x), \quad 0 < x < \infty, \tag{2b}$$

$$q(0,t) = 0, \quad t > 0, \tag{2c}$$

where $q(x,t)$ has sufficient decay for all t as $x \to \infty$ and $q_0(x)$ is a given smooth function with sufficient decay as $x \to \infty$ satisfying $q_0(0) = 0$.

It is often stated that a separable linear PDE in (x,t) formulated in a separable domain can be solved by either a transform in x or a transform in t. For example, the Dirichlet problem of the heat equation on the half-line can be solved either by the sine transform in x or by the Laplace transform in t. The first surprise is that there does *not* exist an *appropriate* x-transform for problem (2). In other words, there does *not* exist an analogue of the x-sine transform for evolution equations involving a third order derivative in x. Of course, problem (2) *can* be analyzed by the Laplace transform in t; however, this approach has several limitations: (a) it involves $\exp[-st + \lambda(s)x]$, where $\lambda(s)$ solves the *cubic* equation $\lambda^3 + \lambda - s = 0$, whereas an x-transform would involve $\exp[ikx - w(k)t]$, where $w(k)$ is the *explicit* expression $ik - ik^3$; (b) it requires t going to ∞, which is *not* natural for an evolution equation (in order to overcome this difficulty, one appeals to causality arguments); and (c) it does *not* generalize to integrable nonlinear PDEs.

Let $\hat{q}(k,t)$ denote the x-Fourier transform of the solution of an evolution equation formulated on the half-line,

$$\hat{q}(k,t) = \int_0^\infty e^{-ikx} q(x,t)dx, \quad \operatorname{Im} k \leq 0, \tag{3}$$

where the restriction $\operatorname{Im} k \leq 0$ is needed in order for $\hat{q}(k,t)$ to make sense. This transform is *not* the appropriate transform for boundary value problems on the half-line for evolution PDEs involving second or higher order derivatives. Consider, for example, the Dirichlet problem of the *heat equation* on the half-line. Differentiating (3) with respect to t, replacing q_t by q_{xx}, and integrating by parts, we find that $\hat{q}(k,t)$ satisfies the equation

$$\hat{q}_t(k,t) + k^2\hat{q}(k,t) + q_x(0,t) + ikq(0,t) = 0, \quad \operatorname{Im} k \leq 0, \tag{4}$$

which involves the unknown Neumann boundary value $q_x(0,t)$. The sine transform is the appropriate transform for this problem, precisely because it eliminates the dependence on $q_x(0,t)$. Similarly, the x-Fourier transform is *not* the appropriate transform for problem (2), because the equation for $\hat{q}(k,t)$ involves the unknown boundary values $q_{xx}(0,t)$ and $q_x(0,t)$,

$$\hat{q}_t(k,t) + (ik - ik^3)\hat{q}(k,t) - q_{xx}(0,t) - ikq_x(0,t) + (k^2-1)q(0,t) = 0, \operatorname{Im} k \leq 0. \tag{5}$$

In summary, the initial-boundary value problem (2) can be analyzed by a Laplace transform in t. This approach has the advantage that it eliminates the dependence on the unknown boundary values $q_{xx}(0,t)$ and $q_x(0,t)$, but it has the disadvantages mentioned earlier. The direct application of the Fourier transform (3) seems to fail because (5) involves the unknown functions $q_x(0,t)$ and $q_{xx}(0,t)$.

It was shown in [1] that by formulating linear evolution PDEs in terms of Lax pairs, it is possible to obtain an integral representation which provides a generalization of the classical Fourier transform. This approach, which is presented in Part III, implies that the solution of (2), with $0 < x < \infty$, $t > 0$, can be represented in the following elegant form:

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx+(ik^3-ik)t}\hat{q}_0(k)dk - \frac{1}{2\pi}\int_{\partial D^+} e^{ikx+(ik^3-ik)t}\tilde{g}(k)dk, \tag{6}$$

where the contour ∂D^+, depicted in Figure 1, is the positively oriented boundary of the domain D^+ defined by

$$D^+ = \{k \in \mathbb{C}, \quad \operatorname{Im} k > 0, \quad \operatorname{Re}[ik^3 - ik] > 0\}, \tag{7}$$

$\hat{q}_0(k)$ is the Fourier transform of $q_0(x)$,

$$\hat{q}_0(k) = \int_0^\infty e^{-ikx} q_0(x)dx, \quad \operatorname{Im} k \leq 0, \tag{8}$$

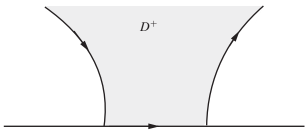


Figure 1. *The contour* ∂D^+ *for equation* (2a) *on the half-line.*

and $\tilde{g}(k)$ is given explicitly in terms of $\hat{q}_0$ evaluated at $\nu_1(k)$ and $\nu_2(k)$,

$$\tilde{g}(k) = \frac{1}{\nu_1 - \nu_2}\left[(\nu_1 - k)\hat{q}_0(\nu_2) + (k - \nu_2)\hat{q}_0(\nu_1)\right], \quad k \in \mathbb{C}, \tag{9a}$$

where ν_1, ν_2 are the two nontrivial functions which leave the associated dispersion relation invariant, i.e.,

$$\nu \neq k, \quad \nu^3 - \nu = k^3 - k, \quad \nu^2 + k\nu + k^2 - 1 = 0. \tag{9b}$$

I.1 A Generalization of the Classical Transforms for Linear Evolution Equations

The representation (6) retains all the advantages of the classical Fourier transform. In particular,

1. it involves explicit exponential dependence on (x, t), and the relevant exponential has an explicit and analytic dependence on k.

2. it provides the *spectral decomposition* of the solution; i.e., the spectral functions $\hat{q}_0(k)$ and $\tilde{g}(k)$ depend only on k and furthermore, these functions are expressed as integrals of the given data.

3. it is uniformly convergent at the boundary.

It is possible to obtain similar integral representations for evolution equations with x-derivatives of *arbitrary* order, formulated either on the half-line or on the finite interval. Although such representations were first derived using the Lax pair approach, it was later understood that these representations can be derived *directly* [2]. A straightforward approach, which uses only Cauchy's theorem and Jordan's lemma, is presented below (see Chapter 1 for details).

I.1.1. The Half-Line

The starting point of the direct approach is the derivation of the equation satisfied by the x-Fourier transform defined in (3). This equation can be derived either by using integration by parts or by rewriting the given evolution equation in an appropriate divergence form and employing Green's theorem (see the discussion in section I.3 below). For the heat equation, after integrating (4) we find

$$e^{k^2 t}\hat{q}(k,t) = \hat{q}_0(k) - ik\int_0^t e^{k^2 s} q(0,s)ds - \int_0^t e^{k^2 s} q_x(0,s)ds, \ \ \mathrm{Im}\, k \leq 0, \tag{10}$$

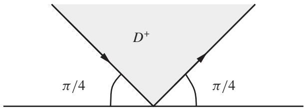


Figure 2. *The contour ∂D^+ for the heat equation on the half-line.*

where $\hat{q}_0(k)$ is the Fourier transform of the initial condition; see (8). Introducing the notations

$$\tilde{g}_0(k,t) = \int_0^t e^{ks} q(0,s)ds, \quad \tilde{g}_1(k,t) = \int_0^t e^{ks} q_x(0,s)ds, \quad k \in \mathbb{C}, \tag{11}$$

equation (10) can be rewritten in the form

$$e^{k^2 t}\hat{q}(k,t) = \hat{q}_0(k) - ik\tilde{g}_0(k^2,t) - \tilde{g}_1(k^2,t), \quad \operatorname{Im} k \le 0. \tag{12}$$

Solving (3) for $q(x,t)$ in terms of $\hat{q}(k,t)$ through the inverse Fourier transform and then replacing $\hat{q}(k,t)$ by the expression obtained from (12), we find

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^2t}\hat{q}_0(k)dk - \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^2t}[ik\tilde{g}_0(k^2,t) + \tilde{g}_1(k^2,t)]dk,$$
$$0 < x < \infty, \quad t > 0. \tag{13}$$

So far we have done nothing novel: equation (13) is the classical representation of the heat equation on the half-line obtained by the x-Fourier transform. As noted earlier, this transform is *not* the appropriate transform for either the Dirichlet ($q(0,t)$ given) or the Neumann ($q_x(0,t)$ given) problem, since (13) contains the t-transforms of both $q(0,t)$ and $q_x(0,t)$.

The new method is based on two novel ideas.

(a) *Deform the contour of integration of the terms involving the boundary values from the real axis to an appropriate contour in the complex k-plane.* For the heat equation this contour, depicted in Figure 2, is the union of the rays $\arg k = \pi/4$ and $\arg k = 3\pi/4$, and can be determined as follows. Let $k = k_R + ik_I$; the real part of ikx equals $-k_I x$, and thus $\exp[ikx]$ is bounded in the upper half complex k-plane, while the real part of $-k^2(t-s)$ equals $-(k_R^2 - k_I^2)(t-s)$, and thus $\exp[-k^2(t-s)]$ is bounded for $k_R^2 \ge k_I^2$ (since $t \ge s$). Hence, under the assumption that $q(0,t)$ and $q_x(0,t)$ are smooth functions, Jordan's lemma, applied in the domain between the real axis and the contour ∂D^+, implies the desired result.

(b) *Utilize the invariant properties of the equation satisfied by $\hat{q}(k,t)$.* For the heat equation this equation is (12), which involves the functions $\tilde{g}_0(k^2,t)$ and $\tilde{g}_1(k^2,t)$. These functions remain invariant if k is replaced by $-k$, and thus we supplement (12) with the equation

$$e^{k^2 t}\hat{q}(-k,t) = \hat{q}_0(-k) + ik\tilde{g}_0(k^2,t) - \tilde{g}_1(k^2,t), \quad \operatorname{Im} k \ge 0. \tag{14}$$

Since this equation is valid for $\operatorname{Im} k \ge 0$, it is valid for k on ∂D^+ (whereas (12) is *not* valid for k on ∂D^+). Equation (13) with the contour in the second integral replaced by ∂D^+ together with (14) immediately yields the solution of the Dirichlet, the Neumann, or the

Robin problem. For example, let $q(x,t)$ solve the Dirichlet problem of the heat equation, i.e.,

$$q_t(x,t) = q_{xx}(x,t), \quad 0 < x < \infty, \quad 0 < t < T, \tag{15a}$$

$$q(x,0) = q_0(x), \quad 0 < x < \infty, \tag{15b}$$

$$q(0,t) = g_0(t), \quad 0 < t < T, \tag{15c}$$

where T is a finite positive constant, $q(x,t)$ has sufficient decay for all t as $x \to \infty$, $q_0(x)$ and $g_0(t)$ are smooth functions satisfying $q_0(0) = g_0(0)$, and $q_0(x)$ has sufficient decay as $x \to \infty$. Then,

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^2t}\hat{q}_0(k)dk - \frac{1}{2\pi}\int_{\partial D^+} e^{ikx-k^2t}\left[\hat{q}_0(-k) + 2ikG_0(k^2,t)\right]dk, \quad 0 < x < \infty, \quad 0 < t < T, \tag{16}$$

where $\hat{q}_0(k)$ is the Fourier transform of $q_0(x)$ defined in (8) and $G_0(k,t)$ is the t-transform of the Dirichlet datum,

$$G_0(k,t) = \int_0^t e^{ks} g_0(s)ds, \quad k \in \mathbb{C}. \tag{17}$$

Indeed, replacing $\tilde{g}_0$ by G_0 in (13) and (14), solving (14) for $\tilde{g}_1$, and substituting the resulting expression in (13), we find (16) plus the following additional term:

$$\frac{1}{2\pi}\int_{\partial D^+} e^{ikx}\hat{q}(-k,t)dk, \quad 0 < x < \infty, \quad t > 0.$$

However, the functions $\exp[ikx]$ and $\hat{q}(-k,t)$ are bounded and analytic for $\operatorname{Im} k > 0$, and thus the application of Cauchy's theorem in the domain of the complex k-plane above ∂D^+ implies that the above term vanishes.

The above derivation is based on the formula for $\hat{q}(k,t)$ and hence it *assumes* that $q(x,t)$ exists. However, it is possible to establish rigorously the validity of (16) *without* the a priori assumption of existence. For the relevant proof see Section 1.5.

I.1.1.1. Derivation of the Classical Representations

The equation satisfied by $\hat{q}(k,t)$, in addition to providing the starting point for the implementation of the new method, also provides a simple alternative to deriving the classical representations. For example, for the heat equation, (12) and (14) are both valid for $k \in \mathbb{R}$. In order to solve the Dirichlet or the Neumann problem we subtract or add these equations. In the Dirichlet case we find

$$\int_0^\infty q(x,t)\sin(kx)dx = e^{-k^2t}\left[\int_0^\infty q_0(x)\sin(kx)dx - kG_0(k^2,t)\right], \quad k \in \mathbb{R}. \tag{18}$$

Hence,

$$q(x,t) = \frac{2}{\pi}\int_0^\infty e^{-k^2t}\sin(kx)\left[\hat{q}_s(k) - kG_0(k^2,t)\right]dk, \quad 0 < x < \infty, \quad t > 0, \tag{19}$$

where $\hat{q}_s$ denotes the sine transform of q_0. The integrand in the right-hand side (RHS) of (19), in contrast to the one given by (16), is *not* uniformly convergent as $x \to 0$. This has both analytical and numerical disadvantages. For example, in order to justify the representation (19) rigorously (without the a priori assumption of existence), one must prove that $q(0,t) = g_0(t)$. Although the analogous verification is elementary for (16) (see Section 1.5), it is *not* straightforward for (19). The advantage of the new method for numerical computations will be discussed below.

Equation (19) can also be derived from (16) by using Cauchy's theorem to deform ∂D^+ back to the real axis.

It must be emphasized that the new method also works for problems (such as the problem formulated by (2)) for which there does *not* exist an appropriate x-transform. We note that it is always possible to deform from the real axis to a contour in the complex k-plane *before* using the invariant properties of the equation for $\hat{q}(k,t)$ (equation (14) for the heat equation) to eliminate the unknown boundary values. But, in general it is *not* possible to deform back to the real axis *after* using the equation for $\hat{q}(k,t)$. Actually, this "return to the real line" is possible only in the exceptional cases that there exists a classical x-transform pair.

I.1.1.2. Numerical Evaluations

The exponential term appearing in the first term of the RHS of (16) oscillates in x and decays in t, while the exponential term appearing in the second term of the RHS of (16) oscillates in t and decays in x. Using appropriate contour deformations it is possible to obtain integrands with decay in both x and t, and this yields an efficient numerical algorithm [3]. Consider for example the initial-boundary value problem (15) with

$$q_0(x) = xe^{-a^2x}, \quad 0 < x < \infty; \quad g_0(t) = \sin bt, \quad t > 0,$$

where a, b are real constants.

Computing $\hat{q}_0(k)$, $G_0(k,t)$ and using contour deformation, (16) yields (see Chapter 3 for details)

$$q(x,t) = \frac{1}{2\pi}\int_{\mathcal{L}} \left\{ e^{ikx-k^2t}\left[\frac{1}{(ik+a^2)^2} - \frac{1}{(ik-a^2)^2}\right] - ke^{ikx}\left[\frac{e^{ibt}-e^{-k^2t}}{k^2+ib} - \frac{e^{-ibt}-e^{-k^2t}}{k^2-ib}\right]\right\} dk,$$

where $\mathcal{L}$, depicted in Figure 3, is any smooth curve between ∂D^+ and the real axis. The integrand of the above integral decays exponentially as $k \to \infty$ on $\mathcal{L}$, and thus $q(x,t)$ can be computed efficiently [3].

I.1.1.3. The Associated Green's Function

Using the integral representation for $q(x,t)$ it is straightforward to compute the associated Green's function. For example, (16) implies

$$q(x,t) = \int_0^\infty G^{(I)}(x,t,\xi)q_0(\xi)d\xi + \int_0^t G^{(B)}(x,t,s)g_0(s)ds, \tag{20}$$

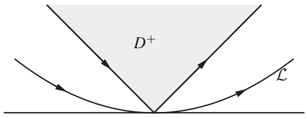


Figure 3. *The contour $\mathcal{L}$ for the heat equation on the half-line.*

where

$$G^{(I)}(x,t,\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-\xi)-k^2t}dk - \frac{1}{2\pi}\int_{\partial D^+} e^{ik(x+\xi)-k^2t}dk,$$

$$0 < x, \xi < \infty, \quad t > 0, \tag{21a}$$

$$G^{(B)}(x,t,s) = -\frac{i}{\pi}\int_{\partial D^+} e^{ikx-k^2(t-s)}dk, \quad 0 < x < \infty, \quad 0 < s < t, \quad t > 0. \tag{21b}$$

It is possible to write similar expressions for an evolution PDE with spatial derivatives of arbitrary order. However, although the integrals appearing in (20) can be computed explicitly, the analogous integrals in the general case cannot be computed in closed form.

I.1.1.4. The Generalized Dirichlet to Neumann Map

For evolution PDEs the main difficulty with boundary, as opposed to initial-value, problems stems from the fact that only a subset of the boundary values is prescribed as boundary conditions. The determination of the unknown boundary values is often called the Dirichlet to Neumann map [4]. Although this terminology is usually used for elliptic problems, the question of determining the Dirichlet to Neumann map, or more precisely the *generalized Dirichlet to Neumann map*, is also important for evolution PDEs. For example, for the Dirichlet problem associated with (2a), an example of such a map is the determination of $\{q_x(0,t), q_{xx}(0,t)\}$ in terms of $\{q(0,t), q(x,0)\}$. This question is important for both analytical and numerical reasons. For example, following the influential work of [5], the construction of such maps provides the basis of the so-called method of reflectionless boundary conditions, which for evolution equations is a method for the numerical computation of the initial-value problem on the infinite line. We note that although there exists an extensive literature using this approach for *second* order evolution PDEs (see, for example, [6], [7]), this approach has not been extensively used for *third* or *higher* order PDEs. This is perhaps the consequence of the nonavailability of a straightforward method for constructing the generalized Dirichlet to Neumann maps for evolution PDEs involving spatial derivatives of higher order.

Using the new method, it is possible to express $q(x,t)$ in terms of $q_0(x)$ and of the given boundary conditions. Hence, by computing an appropriate number of spatial derivatives and by evaluating the resulting expressions at $x = 0$, it is straightforward to determine the unknown boundary values in terms of the given data. Actually, it turns out that it is also possible to determine the unknown boundary values directly *without* having to determine $q(x,t)$.

As an example consider the heat equation and, for the sake of brevity of presentation, let $q_0(x) = 0$. Evaluating (14) at $t = T$ we find that for $\operatorname{Im} k \geq 0$,

$$-\int_0^T e^{k^2 s} q_x(0, s) ds + ik \int_0^T e^{k^2 s} q(0, s) ds = e^{k^2 T} \int_0^\infty e^{ikx} q(x, T) dx. \tag{22}$$

Equation (22) characterizes the *Dirichlet to Neumann correspondence*; i.e., it relates the Dirichlet and Neumann boundary values. Equation (22) involves $q(x, T)$; however, this function does *not* affect the Dirichlet to Neumann correspondence. For example, in order to determine the Neumann to Dirichlet map we must solve (22) for $q(0, t)$ in terms of $q_x(0, t)$. In what follows we will show that this expression is independent of $q(x, T)$:

$$q(0, t) = -\frac{1}{\sqrt{\pi}} \int_0^t \frac{q_x(0, s)}{\sqrt{t - s}} ds, \quad 0 < t < T. \tag{23}$$

Indeed, we multiply (22) by $\exp[-k^2 t]$ and integrate the resulting equation along ∂D^+, which is the contour depicted in Figure 2. The term $\exp[k^2(T - t)]$ is bounded and analytic in the domain above ∂D^+, and the x-Fourier transform of $q(x, T)$ is analytic in the upper half of the complex k-plane and of $0(1/k)$ as $k \to \infty$, thus the RHS of (22) yields a zero contribution. Furthermore, the change of variables $k^2 = il$ and the classical Fourier transform formula imply that the second term on the LHS of (22) yields $-\pi q(0, t)$. Hence, we find

$$\int_{\partial D^+} \left[\int_0^T e^{k^2(s-t)} q_x(0, s) ds \right] dk + \pi q(0, t) = 0.$$

We split the integral $\int_0^T$ into $\int_0^t$ and $\int_t^T$. The second integrand vanishes due to the fact that the integrand is analytic above ∂D^+, whereas the first integral can be simplified as follows:

$$\int_{\partial D^+} \left[\int_0^t e^{-k^2(t-s)} q_x(0, s) ds \right] dk = \int_{\partial D^+} \left[\int_0^t e^{-l^2} \frac{q_x(0, s) ds}{\sqrt{t - s}} \right] dl$$
$$= c \int_0^t \frac{q_x(0, s) ds}{\sqrt{t - s}},$$

where $l = k\sqrt{t - s}$ and

$$c = \int_{\partial D^+} e^{-l^2} dl = \sqrt{\pi}.$$

In a similar way it can be shown (see Section 1.4) that if $q(x, t)$ satisfies the equation

$$q_t + q_{xxx} = 0, \qquad 0 < x < \infty, \qquad t > 0,$$

with $q_0(x) = 0$, then the Dirichlet and Neumann boundary values can be expressed in terms of the second Neumann boundary value by the following formulae:

$$q(0, t) = c_1 \int_0^t \frac{q_{xx}(0, s)}{(t - s)^{\frac{1}{3}}} ds, \quad q_x(0, t) = c_2 \int_0^t \frac{q_{xx}(0, s)}{(t - s)^{\frac{2}{3}}} ds, \quad t > 0,$$

where the constants c_1 and c_2 can be expressed in terms of the Gamma function.

I.1.2. The Finite Interval

It was mentioned earlier that there does *not* exist an analogue of the sine transform in x for the solution of the Dirichlet problem for (2a) formulated on the half-line. Similarly, there does *not* exist an analogue of the sine series of the solution of the following problem formulated on the finite interval:

$$q_t(x,t) + q_x(x,t) + q_{xxx}(x,t) = 0, \quad 0 < x < L, \quad t > 0, \tag{24a}$$

$$q(x,0) = q_0(x), \quad 0 < x < L, \tag{24b}$$

$$q(0,t) = q(L,t) = q_x(L,t) = 0, \quad t > 0, \tag{24c}$$

where L is a finite positive constant and $q_0(x)$ is a smooth function satisfying $q_0(0) = q_0(L) = \dot{q}_0(L) = 0$. Actually, it can be shown that the operator $\partial_x + \partial_x^3$ satisfying the boundary conditions

$$\phi(0) = \phi(L) = \phi'(L) = 0$$

does *not* possess a complete set of discrete eigenfunctions [8]. In other words, in spite of the fact that (24a) is formulated on a finite interval, the solution $q(x,t)$ *cannot* be expressed in the form of an infinite series. On the other hand, the new method yields the following elegant representation:

$$\begin{aligned} q(x,t) = {} & \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx+(ik^3-ik)t}\hat{q}_0(k) - \frac{1}{2\pi}\int_{\partial D^+} e^{ikx+(ik^3-ik)t}\tilde{g}(k)dk \\ & - \frac{1}{2\pi}\int_{\partial D^-} e^{ik(x-L)+(ik^3-ik)t}\tilde{h}(k)dk, \;\; 0 < x < L, \;\; t > 0, \end{aligned} \tag{25}$$

where ∂D^+ is the oriented boundary of the domain D^+ defined in (7), ∂D^-, depicted in Figure 4, is the oriented boundary of the domain D^- defined by

$$D^- = \{k \in \mathbb{C}, \operatorname{Im} k < 0, \operatorname{Re}(ik - ik^3) < 0\}, \tag{26}$$

$\hat{q}_0(k)$ is the Fourier transform of $q_0(x)$,

$$\hat{q}_0(k) = \int_0^L e^{-ikx} q_0(x)dx, \quad k \in \mathbb{C}, \tag{27}$$

and the functions $\tilde{g}(k)$ and $\tilde{h}(k)$ can be expressed explicitly in terms of $\hat{q}_0$ evaluated at $\nu_1(k)$, $\nu_2(k)$ (defined in (9b)):

$$\begin{aligned} \tilde{g}(k) = {} & \frac{1}{\Delta(k)}\Big\{e^{-i\nu_1 L}\big[k\hat{q}_0(\nu_2) - \nu_2\hat{q}_0(k)\big] + e^{-i\nu_2 L}\big[\nu_1\hat{q}_0(k) - k\hat{q}_0(\nu_1)\big] \\ & + e^{-ikL}\big[\nu_2\hat{q}_0(\nu_1) - \nu_1\hat{q}_0(\nu_2)\big]\Big\} \\ & - \frac{k}{\Delta(k)}\Big\{e^{-i\nu_1 L}\big[\hat{q}_0(k) - \hat{q}_0(\nu_2)\big] + e^{-i\nu_2 L}\big[\hat{q}_0(\nu_1) - \hat{q}_0(k)\big] \\ & + e^{-ikL}\big[\hat{q}_0(\nu_2) - \hat{q}_0(\nu_1)\big]\Big\}, \end{aligned} \tag{28a}$$

$$\tilde{h}(k) = \frac{1}{\Delta(k)}\big[(\nu_2 - \nu_1)\hat{q}_0(k) + (\nu_2 - k)\hat{q}_0(\nu_1) + (k - \nu_1)\hat{q}_0(\nu_2)\big], \tag{28b}$$

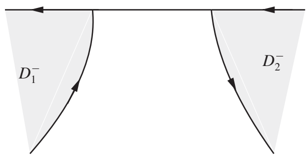


Figure 4. *The contour ∂D^- for* (24a) *on the finite interval.*

where

$$\Delta(k) = e^{-ikL}(\nu_1 - \nu_2) + e^{-i\nu_2 L}(k - \nu_1) + e^{-i\nu_1 L}(\nu_2 - k). \quad (28c)$$

The derivation of integral representations such as (25) follows steps identical to those used for the derivation of the analogous representations for the half-line (see [9], [10], [11]). For example, for the heat equation, the derivation is based on the analogue of (12) and (14), which are the following equations valid for all complex values of k:

$$e^{k^2 t}\hat{q}(k,t) = \hat{q}_0(k) - ik\tilde{g}_0(k^2,t) - \tilde{g}_1(k^2,t) + e^{-ikL}\left[\tilde{h}_1(k^2,t) + ik\tilde{h}_0(k^2,t)\right], \quad (29a)$$

$$e^{k^2 t}\hat{q}(-k,t) = \hat{q}_0(-k) + ik\tilde{g}_0(k^2,t) - \tilde{g}_1(k^2,t) + e^{ikL}\left[\tilde{h}_1(k^2,t) - ik\tilde{h}_0(k^2,t)\right], \quad (29b)$$

where $\hat{q}_0(k)$ is defined by (27), $\hat{q}(k,t)$ is the Fourier transform of $q(x,t)$ (equation (3) with ∞ replaced by L), and $\tilde{h}_0$, $\tilde{h}_1$ denote the t-transforms of $q(L,t)$ and $q_x(L,t)$,

$$\tilde{h}_0(k,t) = \int_0^t e^{ks} q(L,s)ds, \quad \tilde{h}_1(k,t) = \int_0^t e^{ks} q_x(L,s)ds, \quad k \in \mathbb{C}, t > 0. \quad (30)$$

In the same way that the analogue of (12) and (14) yields a novel representation for problems on the half-line, the analysis of (28) yields novel representations for boundary value problems formulated on the finite interval. For example, let $q(x,t)$ solve the Dirichlet problem,

$$q_t(x,t) = q_{xx}(x,t), \quad 0 < x < L, \quad 0 < t < T, \quad (31a)$$

$$q(x,0) = q_0(x), \quad 0 < x < L, \quad (31b)$$

$$q(0,t) = g_0(t), \quad q(L,t) = h_0(t), \quad t > 0, \quad (31c)$$

where L and T are finite positive constants and q_0, g_0, h_0 are smooth functions satisfying $g_0(0) = q_0(0)$, $h_0(0) = q_0(L)$. Then

$$\begin{aligned} q(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^2 t}\hat{q}_0(k)dk \\ &-\frac{1}{2\pi}\int_{\partial D^+} \frac{e^{ikx-k^2 t}}{\Delta(k)}\left[e^{ikL}\hat{q}_0(k) - e^{-ikL}\hat{q}_0(-k) - 2ike^{-ikL}G_0(k^2,t) + 2ikH_0(k^2,t)\right]dk \\ &-\frac{1}{2\pi}\int_{\partial D^-} \frac{e^{ik(x-L)-k^2 t}}{\Delta(k)}\left[\hat{q}_0(k) - \hat{q}_0(-k) - 2ikG_0(k^2,t) + 2ike^{ikL}H_0(k^2,t)\right]dk, \end{aligned}$$

$$0 < x < L, \quad 0 < t < T, \quad (32)$$

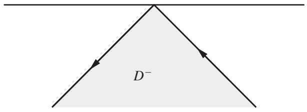


Figure 5. *The curves ∂D^- for the heat equation on the finite interval.*

where ∂D^+ is the union of the rays $\arg k = \pi/4$ and $\arg k = 3\pi/4$ depicted in Figure 3, ∂D^- is the union of the rays $\arg k = -\pi/4$ and $\arg k = -3\pi/4$ depicted in Figure 5, $\hat{q}_0(k)$ and G_0 are defined by (27) and (17), H_0 is defined by

$$H_0(k,t) = \int_0^t e^{ks} h_0(s)ds, \quad k \in \mathbb{C}, \tag{33a}$$

and Δ is given by

$$\Delta(k) = e^{ikL} - e^{-ikL}, \quad k \in \mathbb{C}. \tag{33b}$$

I.1.2.1. Derivation of the Classical Representations

If there exists a classical transform representation, this representation can be obtained in a simple manner using the equations satisfied by $\hat{q}(k,t)$ and the equations obtained through invariance. For example, for the Dirichlet problem of the heat equation, in order to eliminate $\tilde{g}_1(k^2,t)$ we subtract (29), and this yields

$$\begin{aligned} &e^{k^2 t}\int_0^L \left(e^{-ikx} - e^{ikx}\right) q(x,t)dx \\ &= \hat{q}_0(k) - \hat{q}_0(-k) - 2ikG_0(k^2,t) + ik(e^{ikL} + e^{-ikL})H_0(k^2,t) \\ &\qquad - \Delta(k)\tilde{h}_1(k^2,t), \quad k \in \mathbb{C}. \end{aligned} \tag{34}$$

In order to eliminate the unknown function $\tilde{h}_1(k^2,t)$ we evaluate this equation at those values of k for which $\Delta(k) = 0$, i.e. $k = n\pi/L$, $n \in \mathbb{Z}$. This yields an expression for $q(x,t)$ involving the integral with respect to $\sin(n\pi x/L)$, and then $q(x,t)$ follows by inverting this integral.

The main advantage of the above approach for deriving classical representations is that it avoids the derivation of the appropriate transform (in this case the sine series). Furthermore, it also avoids integration by parts.

It appears that the novel *integral* representations obtained by the new method have both analytical and numerical advantages in comparison with the classical *infinite series* representations: (a) They are uniformly convergent at both $x = 0$ and $x = L$. Analytically, this makes it easier to prove rigorously the validity of such representations *without* the a priori assumption of existence. Numerically, using appropriate contour deformations, it makes it possible to obtain integrands which decay exponentially as $k \to \infty$, and this leads to efficient numerical computations. (b) These representations retain their form even for more complicated boundary conditions, whereas the classical representations involve infinite series over a spectrum determined by a transcendental equation. For example, in the case of the heat equation with the Robin boundary conditions,

$$q_x(0,t) - \gamma_1 q(0,t) = h_1(t), \quad q_x(L,t) - \gamma_2 q(L,t) = h_2(t), \quad t > 0,$$
$$\gamma_1 > 0, \gamma_2 > 0, \gamma_1 \neq \gamma_2$$

$q(x,t)$ is given by a formula similar to (32), where $\Delta(k)$ is now given by

$$\Delta(k) = -(ik+\gamma_1)(ik+\gamma_2)e^{ikL} + (ik-\gamma_1)(ik-\gamma_2)e^{-ikL}.$$

On the other hand, the classical representation involves an infinite series over $\{k_n\}_0^\infty$, where k_n satisfy the transcendental equation $\Delta(k_n) = 0$. In spite of the advantage of an *integral* representation involving an *explicit* kernel, as opposed to an *infinite* series representation involving a spectrum satisfying a *transcendental* equation, all standard textbooks present the latter representation.

The classical representations can also be obtained from (32) by using Cauchy's theorem and deforming ∂D^+ and ∂D^- back to the real axis.

I.1.2.2. The Discrete Spectrum

The solution representations (25) and (32), discussed earlier, raise interesting questions about the existence of a discrete spectrum for evolution equations formulated on the finite interval. It appears that there does *not* exist a discrete spectrum for (25), while regarding (32) the existence of a discrete spectrum is a matter of definition. Indeed, the classical sine-series representation is based on the discrete spectrum of the associated self-adjoint operator ∂_x^2. On the other hand, looking for a solution $q(x,t)$ treating the heat equation directly as a PDE (without appealing to the spectral expansion of the associated ODE in x), it is possible to express $q(x,t)$ as an integral, thus avoiding altogether the discrete spectrum of the associated ODE.

I.1.2.3. The Associated Green's Function

Using the integral representation for $q(x,t)$ it is straightforward to compute the associated Green's function. In general, if an initial-boundary value problem involves n_1 and n_2 boundary conditions at $x = 0$ and $x = L$, denoted, respectively, by $\{g_j(t)\}_0^{n_1-1}$ and $\{h_j(t)\}_0^{n_2-1}$, then

$$q(x,t) = \int_0^L G^{(I)}(x,t,\xi)q_0(\xi)d\xi + \sum_{j=0}^{n_1-1} \int_0^t G_j^{(B)}(x,t,s)g_j(s)ds$$
$$+ \sum_{j=0}^{n_2-1} \int_0^t H_j^{(B)}(x,t,s)h_j(s)ds, \quad 0 < x < L, \quad 0 < t < T,$$

where G^I, $\{G_j^B\}_0^{n_1-1}$, $\{H_j\}_0^{n_2-1}$ can be expressed as integrals in the complex k-plane. For example, for the problem defined by equations (31) with $g_0 = h_0 = 0$, the *trace* of $G^{(I)}$, i.e., the function

$$K(t) = \int_0^L G^{(I)}(x,x,t)dx, \quad t > 0,$$

is given by (see Section 2.3 for details)

$$K(t) = \frac{L}{2\sqrt{\pi t}} - \frac{1}{2} - \frac{L}{\pi}\int_{\partial D_0^+} \frac{e^{-k^2 t}}{1 - e^{-2ikL}} dk, \tag{35}$$

where ∂D_0^+ denotes the curve obtained by deforming ∂D^+ to pass above $k = 0$.

For the particular cases that there exists a classical integral representation, $K(t)$ can also be computed in terms of the associated discrete spectrum. For example, for the problem defined by equations (31) with $g_0 = h_0 = 0$,

$$K(t) = \sum_{n=1}^{\infty} e^{-\frac{n^2\pi^2}{L^2}t}. \tag{36}$$

Equations (35) and (36) provide a relation between the classical *spectrum* and the *geometry* of the heat equation [12].

I.1.2.4. Asymptotics

The new method yields elegant integral representations for initial-boundary value problems formulated on the half-line and on a finite integral. These integral representations involve an explicit (x, t) dependence in the form of an exponential with analytic dependence. An important advantage of such representations is that they provide an effective approach to computing the asymptotic properties of the solution. For example, for equations on the half-line it is straightforward to compute the large t asymptotics [13]. Similarly, it is elementary to compute the small t asymptotics. For example, (35) immediately implies the well-known formula

$$K(t) = \frac{1}{2\sqrt{\pi t}} - \frac{1}{2} + O(t^{\infty}).$$

It is straightforward to obtain similar formulae for PDEs involving spatial derivatives of arbitrary order.

I.2 Inversion of Integrals

The classical transform approach to separable boundary value problems for linear PDEs is based on the derivation of an appropriate transform. If such a transform exists, it can be algorithmically derived by constructing the associated Green's function and by integrating this function in the complex λ-plane. For example, for the Dirichlet problem of the heat equation on the half-line, the associated Green's function $G(x, \xi, \lambda)$ satisfies the ODE

$$-G_{xx} - \lambda G = \delta(x - \xi),$$

$$G(0, \xi, \lambda) = 0, \quad \int_0^{\infty} G^2 dx < \infty.$$

Then, the classical formula

$$\delta(x - \xi) = -\lim_{R\to\infty} \int_{|\lambda|=R} G(x, \xi, \lambda) d\lambda \tag{37}$$

yields (see [14])

$$\delta(x-\xi) = \frac{2}{\pi}\int_0^\infty \sin(kx)\sin(k\xi)dk,$$

which defines the sine transform pair.

However, the validity of (37) is based on the assumption that there exists a complete set of eigenfunctions and that $G(x, \xi, \lambda)$ is an analytic function of λ except for poles and branch-point singularities. Under these assumptions the integral in (37) reduces to a sum of residues (the contribution of the discrete spectrum) plus integrals along the branch cuts (the contribution of the continuous spectrum).

The new transform method for linear evolution PDEs, discussed earlier in section I.1, bypasses the method of separation of variables and hence it is *not* based on the construction (or even the existence) of an appropriate transform. The above method can be extended to other types of linear PDEs, such as the basic elliptic equations (see the discussion in section I.4) and hence as far as the new method is concerned the derivation of appropriate transforms is obsolete.

The new method not only does not require classical transforms but actually provides an alternative approach to deriving classical transforms *avoiding* the assumptions of completeness and analyticity. This approach involves (a) expressing q in terms of an integral in the complex k-plane; and (b) using contour deformation and the residue theorem to rewrite q in terms of an infinite series plus integrals along the real axis. This novel approach will be illustrated in Chapter 4.

The above novel technique for constructing classical transforms, although efficient and algorithmic, is conceptually unsatisfactory: Why should one solve a PDE *in two dimensions* in order to construct a transform in *one* dimension? Actually, Israel Gel'fand and the author, motivated from techniques developed in the theory of integrable nonlinear PDEs, introduced in [15] a direct method for constructing integral transform pairs. This method is based on the formulation of either a Riemann–Hilbert (RH) [16] or a d-bar problem [17]. In particular, it was shown in [15] that the *spectral* analysis of the ODE

$$\mu_x(x,k) - ik\mu(x,k) = q(x), \quad x \in \mathbb{R}, \quad k \in \mathbb{C}, \tag{38}$$

yields the classical Fourier transform pair. The first novel integral transform constructed by this new method is the attenuated Radon transform derived by Novikov in [18] (generalizing the analogous derivation of the classical Radon transform presented in [19]). The attenuated Radon transform of a Schwartz function $g(x_1, x_2)$ with attenuation $f(x_1, x_2)$, where f is a given Schwartz function, is defined by

$$\hat{g}_f(\rho,\theta) = \int_{-\infty}^{\infty} e^{-\int_\tau^\infty f(s\cos\theta-\rho\sin\theta,\, s\sin\theta+\rho\cos\theta)ds} \times g(\tau\cos\theta-\rho\sin\theta,\, \tau\sin\theta+\rho\cos\theta)d\tau, \quad \rho\in\mathbb{R}, \quad \theta\in(0,2\pi). \tag{39a}$$

It will be shown in Chapter 7 that the inverse attenuated Radon transform is given by

$$g(x_1,x_2) = \frac{1}{4\pi}\left(\partial_{x_1} - i\partial_{x_2}\right)\int_0^{2\pi} e^{i\theta} J(\rho,\tau,\theta)d\theta, \tag{39b}$$

where (ρ, τ) are given in terms of (x_1, x_2) by the equations

$$\rho = x_2\cos\theta - x_1\sin\theta, \quad \tau = x_2\sin\theta + x_1\cos\theta,$$

and J is defined as follows:

$$J(\rho,\tau,\theta) = e^{\int_\tau^\infty f(s\cos\theta-\rho\sin\theta, s\sin\theta+\rho\cos\theta)ds}$$
$$\times \left(e^{P^-\hat{f}(\rho,\theta)}P^-e^{-P^-\hat{f}(\rho,\theta)} + e^{-P^+\hat{f}(\rho,\theta)}P^+e^{P^-\hat{f}(\rho,\theta)}\right)\hat{g}_f(\rho,\theta),$$
$$(\rho,\tau)\in\mathbb{R}^2, \quad \theta\in(0,2\pi), \tag{39c}$$

where $P^{\pm}$ denote the usual projectors in the variable ρ,

$$(P^{\mp}f)(\rho) = \mp\frac{f}{2} + \frac{1}{2i\pi}\int_{-\infty}^{\infty}\frac{f(\rho')d\rho}{\rho'-\rho}, \tag{39d}$$

and $\hat{f}(\rho,\theta)$ denotes the Radon transform of f which is defined by

$$\hat{f}(\rho,\theta) = \int_{-\infty}^{\infty} f(\tau\cos\theta-\rho\sin\theta, \tau\sin\theta+\rho\cos\theta)d\tau, \quad \rho\in\mathbb{R}, \ \theta\in(0,2\pi). \tag{39e}$$

In the same way that the Radon transform provides the mathematical foundation of computerized tomography (CT), the attenuated Radon transform provides the mathematical foundation of another imaging technique of great medical importance called single photon emission computerized tomography (SPECT).

Another novel application of the ideas of [15] is the introduction in [20] of a methodology for inverting a large class of integrals. An example of such an integral is

$$\hat{f}(k) = \int_0^T e^{-k^2s-ikl(s)}f(s)ds, \quad k\in\mathbb{C}, \tag{40}$$

where $l(s)$ is a given smooth function and T is a finite positive constant. This integral is a variation of the integral

$$\mathcal{F}(k) = \int_0^T e^{-k^2s}f(s)ds, \quad k\in\mathbb{C}, \tag{41}$$

which, as discussed earlier in section I.1.1, appears in the characterization of the Dirichlet to Neumann correspondence of the heat equation in the half-line. Actually, the integral in (40) appears in the analogous problem for the heat equation in the domain $\{l(t) < x < \infty, t > 0\}$. Although the integral (41) can be inverted by a straightforward application of the inverse Fourier transform (after a suitable change of variables), the inversion of (40) is rather complicated. This latter inversion is based on the spectral analysis of the following ODE:

$$\mu_t(t,k) - \left(k^2 + ik\frac{dl(t)}{dt}\right)\mu(t,k) = kf(t), \ \ 0<t<T, \ \ k\in\mathbb{C}.$$

The novelty of this ODE in comparison with (38) is that, for (38) there exists a solution $\mu(x,k)$ which is *sectionally analytic* in k, while for the above equation there does *not* exist such a solution; i.e., $\partial\mu/\partial\bar{k}$ has support in a two-dimensional domain. This makes it necessary to formulate a d-bar as opposed to an RH problem, which in turn implies that the inversion of (40) cannot be written explicitly, but it is characterized through the solution of a linear Volterra integral equation; see Chapter 8 for details. Nevertheless, this equation involves an exponentially decaying kernel, and this leads to efficient numerical computations [21].

I.3 Novel Integral Representations for Linear PDEs

The direct approach to the new method discussed earlier in section I.1 starts with the derivation of the equation satisfied by the Fourier transform of $q(x,t)$ which is denoted by $\hat{q}(k,t)$. The most efficient way of deriving this equation is *not* to use integration by parts as was done in section I.1, but to rewrite the given PDE as *a one-parameter family of PDEs, each of which is in a divergence form.* Let $q(x,t)$ satisfy the linear evolution PDE

$$q_t + w(-i\partial_x)q = 0, \tag{42}$$

where $w(k)$ is a polynomial of degree n such that $\operatorname{Re} w(k) \geq 0$ for k real (this restriction ensures that the initial-value problem of the given PDE is well posed). The PDE (42) admits the one-parameter family of solutions $\exp[ikx - w(k)t]$. It is elementary to show that this PDE can be rewritten in the form

$$\left(e^{-ikx+w(k)t}q(x,t)\right)_t - \left(e^{-ikx+w(k)t}\sum_{j=0}^{n-1} c_j(k)\partial_x^j q(x,t)\right)_x = 0, \quad k \in \mathbb{C}, \tag{43}$$

where $\{c_j(k)\}_0^{n-1}$ can be explicitly computed in terms of $w(k)$; see Chapter 1.

Suppose that the above PDE is valid on the half-line. Then Green's theorem in the domain $\{0 < x < \infty, 0 < s < t\}$ implies the following equation for $\hat{q}(k,t)$:

$$e^{w(k)t}\hat{q}(k,t) = \hat{q}_0(k) - \sum_{j=0}^{n-1} c_j(k)\int_0^t e^{w(k)s}\partial_x^j q(0,s)ds, \quad \operatorname{Im} k \leq 0. \tag{44}$$

Using the inverse Fourier transform and Jordan's lemma, (44) implies that for $0 < x < \infty$, $t > 0$, $q(x,t)$ satisfies

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-w(k)t}\hat{q}_0(k)dk - \frac{1}{2\pi}\int_{\partial D+} e^{ikx-w(k)t}\tilde{g}(k,t)dk, \tag{45}$$

where $\tilde{g}(k,t)$ denotes the summation term appearing in (44) and ∂D^+ is the oriented boundary of D^+,

$$D^+ = \{k \in \mathbb{C}, \quad \operatorname{Im} k > 0, \quad \operatorname{Re} w(k) < 0\}, \tag{46}$$

with the orientation that D^+ is on the left of ∂D^+.

It should be emphasized that the integral representation (45) involves the "global form" of the boundary values $\{\partial_x^j q(0,t)\}_0^{n-1}$, i.e., certain t-integrals of the boundary values. These "global values" are coupled by (44). In what follows we will refer to this equation as the *global relation*.

It was noted in section I.2 that the Fourier transform pair can be derived through the spectral analysis of (38). Is there a spectral interpretation of (45)? The answer is affirmative: It will be shown in Chapter 10 that (45) can also be derived through the *simultaneous* spectral analysis of the following pair of equations called a *Lax pair*:

$$\mu_x(x,t,k) - ik\mu(x,t,k) = q(x,t), \quad 0 < x < \infty, \quad t > 0, \quad k \in \mathbb{C}, \tag{47a}$$

$$\mu_t(x,t,k) + w(k)\mu(x,t,k) = \sum_{j=0}^{n-1} c_j(k)\partial_x^j q(x,t). \tag{47b}$$

Equations (47) are a direct consequence of (43). Indeed, the latter equation motivates the introduction of the potential $M(x,t,k)$ defined by

$$M_x = e^{-ikx+w(k)t}q(x,t), \quad M_t = e^{-ikx+w(k)t}\sum_{j=0}^{n-1} c_j(k)\partial_x^j q(x,t).$$

Letting $M = \mu \exp[-ikx + w(k)t]$, these equations become equations (47).

Equations (47), which are *two* equations for the *single* function μ, are compatible if and only if q satisfies (42).

Letting $q(x,t) = X(x;\lambda)T(t;\lambda)$, (42) yields the two ODEs

$$\frac{dT}{dt} - \lambda T = 0, \quad w\left(-i\frac{d}{dx}\right)X + \lambda X = 0. \tag{48}$$

Comparing (47) with (48), it becomes evident that the former equations express a *deeper form of separability*. Indeed, (47) are *equivalent* to (42), where (48) characterize only the separable class of solutions. Furthermore, the second of equations (48) is an ODE of order n, whereas (47) and (47) are both ODEs of first order. It is puzzling that in spite of these advantages, the Lax pair formulation of linear PDEs does not appear in the classical literature.

Equations (47) can be rewritten in the form

$$d\left[e^{-ikx+w(k)t}\mu\right] = e^{-ikx+w(k)t}\left[q\,dx + \sum_{j=0}^{n-1} c_j(k)\partial_x^j q(x,t)dt\right]. \tag{49}$$

Hence, performing the simultaneous spectral analysis of the Lax pair (47) is equivalent to performing the *spectral analysis of the differential form* (49); see [22].

Lax pairs for linear evolution PDEs were first introduced in [15]. It was later realized that Lax pairs also exist for a large class of linear PDEs which include PDEs with constant coefficients, such as the Laplace, the Helmholtz, and the modified Helmholtz equations. It appears that for the latter basic elliptic equations, the simultaneous spectral analysis of the associated Lax pairs, or equivalently the spectral analysis of the associated differential form, provides the *simplest* way of constructing integral representations in the complex k-plane. For example, the following result will be derived in Section 11.2 (see [22], [25]).

Proposition 1. Let Ω be the interior of a convex bounded polygon in the complex z-plane, with corners $z_1, \ldots, z_n, z_{n+1} = z_1$; see Figure 6. Assume that there exists a solution $q(z,\bar{z})$ of the modified Helmholtz equation

$$q_{z\bar{z}} - \beta^2 q = 0, \quad z \in \Omega, \quad \beta > 0, \tag{50}$$

valid in the interior of Ω and suppose that this solution has sufficient smoothness all the way to the boundary of the polygon. Then q can be expressed in the form

$$q(z,\bar{z}) = \frac{1}{4i\pi}\sum_{j=1}^{n}\int_{l_j} e^{i\beta(kz-\frac{\bar{z}}{k})}\hat{q}_j(k)\frac{dk}{k}, \quad z \in \Omega, \tag{51a}$$

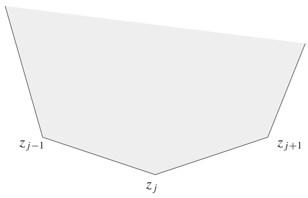


Figure 6. *Part of the polygon.*

where the functions $\{\hat{q}_j(k)\}_1^n$ are defined by

$$\hat{q}_j(k) = \int_{z_j}^{z_{j+1}} e^{-i\beta(kz-\frac{\bar{z}}{k})} \left[(q_z + ik\beta q)\frac{dz}{ds} - \left(q_{\bar{z}} + \frac{\beta}{ik} q \right) \frac{d\bar{z}}{ds} \right] ds, \quad k \in \mathbb{C},$$

$$j = 1, \ldots, n, \quad z_{n+1} = z_1, \tag{51b}$$

$z(s)$ is a parametrization of the side (z_j, z_{j+1}), and $\{l_j\}_1^n$ are the rays on the complex k-plane oriented toward infinity and defined by

$$l_j = \{k \in \mathbb{C} : \arg(k) = -\arg(z_{j+1} - z_j)\}, \quad j = 1, \ldots, n, z_{n+1} = z_1. \tag{51c}$$

Equation (51b) can be written in the form

$$\hat{q}_j(k) = \int_{z_j}^{z_{j+1}} e^{-i\beta(kz-\frac{\bar{z}}{k})} \left[iq_n + i\beta \left(\frac{1}{k}\frac{d\bar{z}}{ds} + k\frac{dz}{ds} \right) q \right] ds, \quad k \in \mathbb{C},$$

$$j = 1, \ldots, n, \quad z_{n+1} = z_1, \tag{51d}$$

where q_n denotes the derivative of q normal to the boundary of Ω.

Furthermore, the following *global relations* are valid:

$$\sum_{j=1}^{n} \hat{q}_j(k) = 0, \quad \sum_{j=1}^{n} \tilde{q}_j(k) = 0, \quad k \in \mathbb{C}, \tag{52}$$

where $\{\tilde{q}_j(k)\}_1^n$ are defined by

$$\tilde{q}_j(k) = \int_{z_j}^{z_{j+1}} e^{i\beta(k\bar{z}-\frac{z}{k})} \left[iq_n + i\beta \left(\frac{1}{k}\frac{dz}{ds} + k\frac{d\bar{z}}{ds} \right) q \right] ds, \quad k \in \mathbb{C},$$

$$j = 1, \ldots, n, z_{n+1} = z_1. \tag{53}$$

The global relations couple the functions q and q_n on the boundary; i.e., they couple the Dirichlet and Neumann boundary values, and thus they characterize the Dirichlet to Neumann correspondence.

I.4 Green's Identities, Images, Transforms, and the Wiener–Hopf Technique: A Unification

There exists a variety of methods for the exact analysis of boundary value problems for linear PDEs in two dimensions. In what follows an attempt will be made to show that the approach presented in this book unifies and generalizes these methods.

We propose that analytical techniques in two dimensions can be divided into two large categories: One is formulated in the *physical plane* (the complex z-plane) and the other is formulated in the *spectral or Fourier plane* (the complex k-plane). The first unified feature of these categories is that they are both based on rewriting the given PDE as a *family of divergent forms*. Indeed, it is well known that the classical Green's identities, which provide the essence of the physical plane formulation, are based on a divergence formulation. Furthermore, the new method starts with rewriting a given PDE as a one-parameter family of PDEs, each of which is in a divergence form (for example, (42) is rewritten in the form (43)).

A given PDE can be rewritten in a divergence form by utilizing the formal adjoint. For example, let q satisfy the modified Helmholtz equation (50),

$$q_{xx} + q_{yy} - 4\beta^2 q = 0, \quad (x, y) \in \Omega, \tag{54}$$

where Ω is a piecewise smooth domain in $\mathbb{R}^2$. Denote by $\tilde{q}$ a solution of the adjoint PDE, which in this case coincides with (54). The equations for q and $\tilde{q}$ imply

$$(\tilde{q}q_x - \tilde{q}_x q)_x - \left(q\tilde{q}_y - q_y\tilde{q}\right)_y = 0, \quad (x, y) \in \Omega. \tag{55}$$

This equation is the starting point of both the physical plane and the spectral plane formulations.

(a) Choose, instead of $\tilde{q}$, the fundamental solution $G(x', y'; x, y)$. Then an analogue of (55) and Green's theorem imply

$$\int_{\partial\Omega} \left[\left(Gq_{y'} - G_{y'}q\right) dx' - \left(Gq_{x'} - G_{x'}q\right) dy'\right] = \begin{cases} q(x, y), (x, y) \in \Omega, & \text{(56a)} \\ 0, (x, y) \notin \Omega, & \text{(56b)} \end{cases}$$

where $\partial\Omega$ denotes the boundary of Ω.

Equation (56b) couples the Dirichlet and the Neumann boundary values; thus it characterizes the *Dirichlet to Neumann correspondence in the physical space*. Equation (56a) provides the *integral representation in the physical space.*

(b) Separation of variables shows that we can choose $\tilde{q} = \exp[k_1 x + k_2 y]$, where $k_1^2 + k_2^2 = 4\beta^2$. Hence, introducing the potential M, (55) implies

$$M_y = e^{k_1 x + k_2 y}(q_x - k_1 q), \quad M_x = e^{k_1 x + k_2 y}(k_2 q - q_y); \tag{57}$$

i.e., M satisfies

$$dM = e^{k_1 x + k_2 y}\left[(k_2 q - q_y)dx + (q_x - k_1 q)dy\right]. \tag{58}$$

Using the parametrization $k_1 = 2\beta \sin\lambda$, $k_2 = 2\beta\cos\lambda$ and letting $\exp[i\lambda] = k$, $M = \mu\exp[k_1 x + k_2 y]$, equations (57) yield the following Lax pair:

$$\mu_y + \beta\left(\frac{1}{k} + k\right)\mu = q_x + i\beta\left(k - \frac{1}{k}\right)q,$$

$$\mu_x + i\beta\left(\frac{1}{k} - k\right)\mu = -q_y + \beta\left(k + \frac{1}{k}\right)q, \quad (x, y) \in \Omega, \quad k \in \mathbb{C}.$$

These equations are equivalent (compare with (58)) with

$$\begin{aligned} d&\left[\mu e^{i\beta(\frac{1}{k}-k)x+\beta(\frac{1}{k}+k)y}\right] \\ &= e^{i\beta(\frac{1}{k}-k)x+\beta(\frac{1}{k}+k)y}\left\{\left[q_x + i\beta\left(k-\frac{1}{k}\right)q\right]dy\right. \\ &\qquad \left. + \left[-q_y + \beta\left(k + \frac{1}{k}\right)q\right]dx\right\}, (x, y) \in \Omega, \quad k \in \mathbb{C}. \end{aligned} \tag{59}$$

Applying Green's theorem in the domain Ω, (59) yields

$$\begin{aligned} \int_{\partial\Omega} & e^{i\beta(\frac{1}{k}-k)x+\beta(\frac{1}{k}+k)y}\left\{\left[q_x + i\beta\left(k-\frac{1}{k}\right)q\right]dy\right. \\ & \left. + \left[-q_y + \beta\left(k+\frac{1}{k}\right)q\right]dx\right\} = 0, \quad k \in \mathbb{C}. \end{aligned} \tag{60}$$

This equation is the *global relation*; i.e., it characterizes the *Dirichlet to Neumann correspondence in the spectral space*. Furthermore, the spectral analysis of the differential form (59) yields the *integral representation in spectral space*. For the case when Ω is a convex polygon, this representation is given in Proposition 1; see equations (51).

I.4.1. From the Physical to the Spectral Plane

The above discussion indicates that the integral representation in the spectral plane is the analogue of the classical integral representation in the physical plane obtained via the fundamental solution (and similarly for the associated global relation). This implies that it should be possible to construct the spectral representation starting with the classical one. This was actually implemented in [26]: Consider for example the modified Helmholtz equation. Representing the product of δ functions in terms of Fourier integrals,

$$\delta(x - x')\delta(y - y') = \frac{1}{4\pi^2}\iint_{\mathbb{R}^2} e^{ik_1(x-x')+ik_2(y-y')}dk_1 dk_2, \tag{61}$$

we see that the fundamental solution of (54) is given by

$$G = -\frac{1}{4\pi^2}\iint_{\mathbb{R}^2} \frac{e^{ik_1(x-x')+ik_2(y-y')}}{k_1^2 + k_2^2 + 4\beta^2}dk_1 dk_2.$$

Substituting this expression in the LHS of (56a) and comparing the resulting equation with the formula for q given by (51a) (the representation in the spectral plane), we observe that the former equation involves integrations with respect to both dk_1 and dk_2, whereas the latter equation involves integration only with respect to dk. It is shown in [26] that after performing a change of variables from (k_1, k_2) to (k_T, k_N), where k_T and k_N are tangent and normal to $\partial\Omega$, it is possible to compute explicitly the integral with respect to dk_N, and then the integral representation in the physical plane yields the integral representation in the spectral plane.

Although the derivation sketched above is more complicated than the derivation via the spectral analysis, it has the advantage that it establishes the equivalence of the representations in the physical and spectral planes.

I.4.2. The Method of Images Revisited

For the evolution equation (42) the global relation and the integral representation in the spectral plane are (44) and (45), respectively. For the implementation of the new method to (42), a crucial role is played by the invariance properties of the global relation (44). This motivates the following question: Is it possible to utilize the invariant properties of the global relation in the physical plane? The answer is affirmative and this provides an alternative and simpler approach to the classical method of images.

In order to illustrate this alternative approach we consider Laplace's equation and look for a representation of q_z instead of q. Laplace's equation can be written in the form $q_{z\bar{z}} = 0$, i.e., $(q_z)_{\bar{z}} = 0$, which shows that q_z is an analytic function. Hence, the integral representation and the global relation for the function q_z are given by

$$\frac{1}{2i\pi}\int_{\partial\Omega}\frac{q_\zeta d\zeta}{\zeta - z} = \begin{cases} q_z, & z \in \Omega, \\ 0, & z \notin \Omega. \end{cases} \tag{62}$$

Let the real-valued function q satisfy Laplace's equation in the upper half complex z-plane with the following oblique Neumann boundary condition:

$$q_y(x, 0)\sin\alpha + q_x(x, 0)\cos\alpha = g(x), \quad -\infty < x < \infty, \tag{63}$$

where $g(x)$ is a given function with sufficient smoothness and decay and $0 < \alpha < \pi/2$. The boundary condition (63) prescribes the derivative of q in a direction making an angle $-\alpha$ with the x-axis. We will show that q_z is given by (see [27])

$$q_z = \frac{e^{i\alpha}}{2i\pi}\int_{-\infty}^{\infty}\frac{g(\xi)d\xi}{\xi - z}, \quad -\infty < x < \infty, \quad y > 0. \tag{64}$$

Indeed, if Ω is the upper half complex z-plane, equations (62) with $z = x + iy$, $\zeta = \xi + i\eta$ become

$$\frac{1}{4i\pi}\int\frac{q_\xi(\xi, 0) - iq_y(\xi, 0)}{\xi - z}d\xi = \begin{cases} q_z, & x \in \mathbb{R}, y > 0, \quad (65a) \\ 0, & x \in \mathbb{R}, y < 0. \quad (65b) \end{cases}$$

The crucial observation is that the operation of complex conjugation followed by the substitution $y \to -y$ leaves $\xi - z$ *invariant*. Hence, performing these operations in (65b), we find

$$-\frac{1}{4i\pi}\int_{-\infty}^{\infty}\frac{q_\xi(\xi,0)+iq_y(\xi,0)}{\xi - z}d\xi = 0, \quad x \in \mathbb{R}, \quad y > 0. \tag{65c}$$

By manipulating (65a) and (65c) it is possible to eliminate the unknown boundary values: Multiplying (65a) and (65c) by $\exp[-i\alpha/2]$ and $-\exp[i\alpha/2]$, respectively, and then adding the resulting equations, we find (64).

Comparing the integral representation obtained by the method of images with the integral representation in the spectral plane, we note that (64) involves *one* integral, whereas the representation in the spectral plane involves *two* integrals (an integral of the boundary data in the physical plane and an integral in the spectral plane). For some boundary value problems it is possible to compute *explicitly* the integral in the spectral plane. These are precisely the problems that can be solved by a finite number of images.

I.4.3. The Wiener–Hopf Technique and the Riemann–Hilbert Formulation

A large class of boundary value problems can be analyzed by the ingenious Wiener–Hopf technique [28]. For such problems there does *not* exist an appropriate transform; hence one uses *some* transform such as the Fourier transform, and then one formulates a Wiener–Hopf equation in the complex continuation of the transform variable. It turns out that *the global relation in the spectral plane and the equations obtained using the invariant properties of this equation provide a generalization of the Wiener–Hopf formulation*. Indeed, in the case of very simple domains these equations yield *directly* the relevant Wiener–Hopf equation. For more complicated domains one obtains a matrix RH problem (we recall that a Wiener–Hopf equation is a particular case of an RH problem). An illustrative example will be discussed in Chapter 13.

I.4.4. A New Transform Method

The application of the new method to evolution PDEs was discussed earlier in section I.1. We recall that this method involves two novel steps: (a) Construct an integral representation in the complex k-plane and derive the associated global relation (for the half-line these are (45) and (44), respectively). (b) By using the invariant properties of the global relation, eliminate the unknown boundary values from the expression in (a).

It turns out that, in addition to evolution PDEs, it is also possible to construct the integral representation and the global relation in the spectral plane for a large class of boundary value problems. Hence for all such problems it is possible to implement the new method (see [22], [23], [24], [25], [29], [30], [31], [32], [33], [34], [35], [36], [37], [38]). For example, for the modified Helmholtz equation in a convex polygon the relevant equations are (51) and (52).

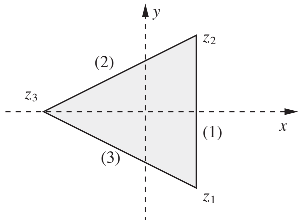


Figure 7. *The equilateral triangle.*

As an illustrative example let us consider the Dirichlet problem for the modified Helmholtz equation in the interior of the equilateral triangle (see Figure 7) with vertices

$$z_1 = \frac{l}{\sqrt{3}} e^{-\frac{i\pi}{3}}, \quad z_2 = \overline{z_1}, \quad z_3 = -\frac{l}{\sqrt{3}}.$$

It turns out that, without loss of generality (see Section 12.5), it is sufficient to consider the symmetric problem, i.e., the problem with the *same* real-valued function $d(s)$ prescribed on each side.

For the sides (1), (2), (3) the following parametrizations will be used:

$$z(s) = \frac{l}{2\sqrt{3}} + is, \;\; z(s) = \left(\frac{l}{2\sqrt{3}} + is\right)\alpha, \;\; z(s) = \left(\frac{l}{2\sqrt{3}} + is\right)\bar{\alpha}, \tag{66}$$

$$-\frac{l}{2} \le s \le \frac{l}{2}, \quad \alpha = e^{\frac{2i\pi}{3}}.$$

Hence, the spectral functions $\{\hat{q}_j(k)\}_1^3$ are defined as follows (see (51d)):

$$\hat{q}_1(k) = E(-ik)\left[iU(k) + \beta\left(\frac{1}{k} - k\right)D(k)\right],$$

$$\hat{q}_2(k) = \hat{q}_1(\alpha k), \quad \hat{q}_3(k) = \hat{q}_1(\bar{\alpha}k), \tag{67}$$

where

$$E(k) = e^{\beta(k+\frac{1}{k})\frac{l}{2\sqrt{3}}}, \quad D(k) = \int_{-\frac{l}{2}}^{\frac{l}{2}} e^{\beta(k+\frac{1}{k})s} d(s)ds,$$

$$U(k) = \int_{-\frac{l}{2}}^{\frac{l}{2}} e^{\beta(k+\frac{1}{k})s} q_n(s)ds, \quad k \in \mathbb{C}. \tag{68}$$

The function $D(k)$ is known, whereas the unknown function $U(k)$ contains the unknown Neumann boundary value q_n.

The reality of $d(s)$ implies that q is real, and hence $\tilde{q}_j(k) = \overline{\hat{q}_j(\bar{k})}$. Thus in this case the global relations (52) become

$$\sum_{j=1}^{3} \hat{q}_j(k) = 0, \quad \sum_{j=1}^{3} \overline{\hat{q}_j(\bar{k})} = 0, \quad k \in \mathbb{C}. \tag{69}$$

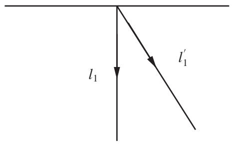


Figure 8. *The contours of integration for the modified Helmholtz equation in the interior on an equilateral triangle.*

The integral representation (51a) involves the integrals of $\{\hat{q}_j(k)\}_1^3$ along the rays $\{l_j\}_1^3$ defined as follows (see (51c)):

$$l_1 = \left\{k \in \mathbb{C}, \ \ \arg k = -\frac{\pi}{2}\right\}, \ \ l_2 = \left\{k \in \mathbb{C}, \arg k = \frac{5\pi}{6}\right\},$$

$$l_3 = \left\{k \in \mathbb{C}, \arg k = \frac{\pi}{6}\right\}. \tag{70}$$

It turns out that by employing the global relations and by using appropriate contour deformations it is possible to eliminate the unknown functions $U(k)$, $U(\alpha k)$, $U(\bar{\alpha} k)$ from (51a); see Section 12.5 for details. This yields the following integral representation (see [23], [32]):

$$q = \frac{1}{4i\pi} \int_{l_1} \left\{ A(k, z, \bar{z}) E(-ik) \left[\beta \left(\frac{1}{k} - k \right) D(k) + \beta \frac{G(k)}{\Delta(\alpha k)} \right] \right\} \frac{dk}{k}$$
$$+ \frac{1}{4\pi i} \int_{l_1'} A(k, z, \bar{z}) E^2(i\alpha k) \beta \frac{G(k)}{\Delta(\alpha k)\Delta(k)} \frac{dk}{k}, \tag{71a}$$

where

$$A = e^{i\beta(kz - \frac{\bar{z}}{k})} + e^{i\beta(\bar{\alpha}k - \frac{\bar{z}}{\bar{\alpha}k})} + e^{i\beta(\alpha k - \frac{\bar{z}}{\alpha k})}, \tag{71b}$$

$$G(k) = \left[\Delta^+(\bar{\alpha}k) \left(\frac{1}{k} - k \right) D(k) + 2 \left(\frac{1}{\bar{\alpha}k} - \bar{\alpha}k \right) D(\bar{\alpha}k) \right.$$
$$\left. + \Delta^+(k) \left(\frac{1}{\alpha k} - \alpha k \right) D(\alpha k) \right], \tag{71c}$$

$$\Delta(k) = e(k) - e(-k), \ \ \Delta^+(k) = e(k) + e(-k), \ \ e(k) = e^{\frac{\beta l}{2}(k + \frac{1}{k})}, \tag{71d}$$

and l_1' is a ray directed toward infinity such that $-\pi/2 < \arg k < -\pi/6$, see Figure 8.

The remarkable feature of the above integral representation is that all relevant integrands decay exponentially. In particular this leads to effective numerical computations.

I.4.5. A Unification and a Novel Numerical Technique

Our current understanding of analytical methods for two-dimensional PDEs can be summarized as follows: Using the formal adjoint it is always possible to rewrite a given PDE in a divergence form. By employing the associated fundamental solution, this formulation immediately yields the *classical Green's representation* as well as the global relation in the physical plane. The algebraic manipulation of the integral representation and of the equations obtained from the global relation through certain invariant transformations provides a simple alternative to the *classical method of images*. On the other hand, the divergence form of a given PDE also yields a Lax pair formulation. This immediately implies the global relation in the spectral plane. If there exists a *classical transform representation,* this representation can be obtained in a simple way by employing the global relation and its invariant consequences, as well as by using the associated inverse transform. In the general case (which includes the case when there does *not* exist an appropriate transform), it is still possible to construct an integral representation in the complex k-plane in terms of certain integrals of the boundary values along the boundary of the domain. This can be achieved by a variety of methods which include the simultaneous spectral analysis of the associated Lax pair. For a large class of boundary value problems it is possible to eliminate the integrals of the unknown boundary values by using the global relation and its invariant consequences. This yields an integral representation in the complex k-plane involving only integrals of the known boundary conditions; hence this method can be considered as a *generalized transform method*. For more complicated problems, it is necessary to determine the unknown boundary values themselves (as opposed to their integrals). For a subclass of these problems this can be achieved through the formulation of a matrix RH problem, which provides a generalization of the classical *Wiener–Hopf technique*.

In the new formulation of the method of images, one starts with the integral representation in the complex z-plane and then eliminates directly the unknown boundary values *by utilizing the invariant properties of the associated global relation*. Similarly, in the generalized transform method, one starts with the integral representation in the complex k-plane and then eliminates the integrals of the unknown boundary values *by utilizing the invariant properties of the associated global relation*. In this sense, the new transform method provides the analogue in the spectral plane of the new formulation of the method of images (which is formulated in the physical plane).

The above discussion is summarized in the diagram below, where q denotes the solution and BVs denotes boundary values.

As was discussed earlier, the main difficulty of boundary value problems stems from the fact that some of the boundary values are *not* prescribed as boundary conditions. For example, for the basic elliptic equations, either the Dirichlet or the Neumann boundary values are unknown. The fundamental importance of the global relation follows from the fact that it couples the boundary values, and therefore it characterizes the generalized Dirichlet to Neumann map.

For elliptic PDEs there exists a well-known method, called the *boundary element method*, which computes numerically the Dirichlet to Neumann map. This method is based on the integral representation of the solution in the physical plane. For the Neumann problem, the limit of this equation, as z approaches the boundary *within* the given domain, yields a linear integral equation for the unknown Dirichlet boundary value (similar considerations are valid for the Dirichlet problem). It must be emphasized that this integral equation can also be obtained by taking the limit of the global relation as z approaches the boundary with z *outside* the given domain. Hence, *the boundary element method is based on the numerical solution of the global relation in the physical plane.* In [39], [40], [41], [42] a numerical method is presented for computing the Dirichlet to Neumann map which is based on the global relation in the spectral plane. This novel method will be discussed in Chapter 14.

Divergence Form

$\swarrow \qquad \searrow$

Physical Plane (z)	Spectral Plane (k)
q in terms of BVs	q in terms of integrals of BVs
global relation	global relation
Method of Images	New Transform Method
eliminate unknown BVs	eliminate integrals of unknown BVs

I.5 Nonlinearization of the Formulation in the Spectral Plane

As it was discussed earlier in section I.4, the solution of a linear PDE can be expressed through appropriate integral representations in both the physical and the spectral plane. Furthermore, the latter representation can be constructed through a variety of methods. It appears that only the integral representation in the spectral plane, and in particular the derivation of this representation through the simultaneous spectral analysis of the Lax pair, can be generalized to integrable nonlinear PDEs. In order to illustrate the "nonlinearization" of the linear approach, we will concentrate on the linear PDE

$$iq_t + q_{xx} = 0. \tag{72}$$

I.5.1. From Linear to Integrable Nonlinear PDEs

Equation (72) is the compatibility condition of the following two equations satisfied by the scalar function $\mu(x, t, k)$:

$$\mu_x - ik\mu = q, \quad \mu_t + ik^2\mu = iq_x - kq, \quad k \in \mathbb{C}. \tag{73}$$

The Cauchy problem of (72) can be easily solved through the Fourier transform in x. On the other hand, it is shown in Chapter 6 that the Fourier transform can be rederived through the spectral analysis of the *first* of equations (73). This leads to the formulation of an RH problem with a "jump" across the real axis of the complex k-plane; this jump is proportional to the Fourier transform of $q(x, t)$ denoted by $\hat{q}(k, t)$. The time evolution of $\hat{q}(k, t)$ can be determined from the *second* of equations (73) (or from the PDE (72) itself). Thus, the solution of the Cauchy problem with the initial condition $q_0(x)$ can be formulated as follows:

$$q(x, t) = -i \lim_{k\to\infty} [k\mu(x, t, k)], \quad x \in \mathbb{R}, \quad t > 0, \tag{74}$$

where μ is a sectionally analytic function in the entire complex k-plane for all $x \in \mathbb{R}$ and $t > 0$, with a jump across the real k-axis,

$$\mu = \begin{cases} \mu^+, & \operatorname{Im} k \geq 0, \\ \mu^-, & \operatorname{Im} k \leq 0, \end{cases}$$

$$\mu^+(x,t,k) - \mu^-(x,t,k) = e^{ikx-ik^2t}\hat{q}_0(k), \quad k \in \mathbb{R}, \tag{75}$$

where $\hat{q}_0(k)$ denotes the Fourier transform of $q_0(x)$.

We will now show that starting with the above RH problem, it is possible to construct a Lax pair for a nonlinear version of (72); see [43], [44]. We start by rewriting the scalar RH problem as the following triangular matrix RH problem for the sectionally analytic 2×2 matrix $M(x,t,k)$, $x \in \mathbb{R}$, $t > 0$,

$$M^+(x,t,k) = M^-(x,t,k)\begin{pmatrix} 1 & e^{ikx-ik^2t}\hat{q}_0(k) \\ 0 & 1 \end{pmatrix}, \quad k \in \mathbb{R}, \tag{76a}$$

$$M(x,t,k) = \operatorname{diag}(1,1) + O\left(\frac{1}{k}\right), \qquad k \to \infty. \tag{76b}$$

Indeed, if M_1 and M_2 denote the column vectors of M, the first column of (76) implies

$$M_1^+ = M_1^-, \quad k \in \mathbb{R}; \quad M_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + O\left(\frac{1}{k}\right), \quad k \to \infty.$$

Thus

$$M_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Then the second column of (76) yields

$$M_2^+ - M_2^- = \begin{pmatrix} e^{ikx-ik^2t}\hat{q}_0(k) \\ 0 \end{pmatrix}, \quad k \in \mathbb{R}; \quad M_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad k \to \infty.$$

Thus

$$M_2 = \begin{pmatrix} \mu \\ 1 \end{pmatrix},$$

where μ satisfies (75) and $\mu = 0(1/k)$ as $k \to \infty$.

We next eliminate the restriction of the triangularity of the jump matrix in (76a), but we retain the unimodular property of this matrix. This yields

$$M^+(x,t,k) = M^-(x,t,k)\begin{pmatrix} 1 & e^{ikx-ik^2t}\rho_1(k) \\ e^{-ikx+ik^2t}\rho_2(k) & 1+\rho_1(k)\rho_2(k) \end{pmatrix}, \quad k \in \mathbb{R}, \tag{77}$$

where ρ_1 and ρ_2 are some functions of k.

Starting from the RH problem defined by (76b) and (77) and employing the powerful *dressing method* introduced by Zakharov and Shabat [45], [46] (see also [47]), it is possible to construct *algorithmically* the following Lax pair (see Chapter 15 for details):

$$M_x + ik[\sigma_3, M] = QM, \tag{78a}$$

$$M_t + 2ik^2[\sigma_3, M] = (2kQ - iQ_x\sigma_3 - iQ^2\sigma_3)M, \tag{78b}$$

where [,] denotes the usual matrix commutator and

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & q(x,t) \\ r(x,t) & 0 \end{pmatrix}. \tag{79}$$

The compatibility condition of (78) yields the following pair of nonlinear evolution PDEs for $q(x, t)$ and $r(x, t)$:

$$\begin{aligned} iq_t + q_{xx} - 2rq^2 = 0, \\ -ir_t + r_{xx} - 2r^2q = 0. \end{aligned} \tag{80}$$

The reduction $r = \lambda\bar{q}, \lambda = \pm 1$, yields the celebrated Nonlinear Schrödinger (NLS) equation (see [48])

$$iq_t + q_{xx} - 2\lambda|q|^2q = 0, \quad \lambda = \pm 1. \tag{81}$$

In summary, we propose the following approach for constructing an integrable nonlinear PDE corresponding to a given linear PDE.

1. Construct a Lax pair for the given linear PDE (for (72), a Lax pair is given by (73)). A Lax pair for an arbitrary linear PDE with constant coefficients is algorithmically constructed in Chapter 9.

2. By analyzing this Lax pair, express the solution of the initial-value problem of the given linear PDE in terms of a scalar RH problem (see (74) and (75)). For an arbitrary linear PDE this analysis is performed in Chapter 10.

3. Rewrite the relevant RH problem in a triangular matrix form (see (76a)), and then "deform" this problem in order to obtain a genuine matrix RH problem; see (77).

4. Starting with the latter problem and by employing the dressing method, construct the associated Lax pair. The compatibility condition of this Lax pair yields a nonlinear integrable PDE.

I.5.2. Simultaneous Spectral Analysis and Integral Representations

Suppose that there exists a smooth solution $q(x, t)$ of the linear PDE (72) formulated on the domain Ω,

$$\Omega = \{0 < x < \infty,\ 0 < t < T\}, \quad T > 0, \tag{82}$$

and suppose that this solution has sufficient decay as $x \to \infty$ for all $0 < t < T$. Then, according to the new transform method, this solution can be expressed in the following form (see (45)) for $(x, t) \in \Omega$:

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-ik^2t}\hat{q}_0(k)dk - \frac{1}{2\pi}\int_{\partial D^+} e^{ikx-ik^2t}\tilde{g}(k)dk, \tag{83}$$

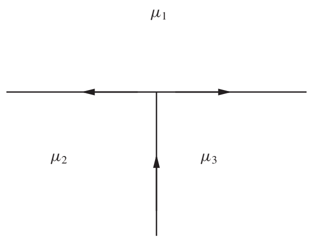


Figure 9. *The function* μ *for* (72).

where ∂D^+ is the oriented boundary of the first quadrant of the complex k-plane (see (46)), $\hat{q}_0(k)$ is the Fourier transform of the initial condition $q_0(x)$, and $\tilde{g}(k)$ is defined in terms of the boundary values $q(0,t)$ and $q_x(0,t)$,

$$\tilde{g}(k) = \int_0^T e^{ik^2 s}\,[iq_x(0,s) + kq(0,s)]\,ds, \quad k \in \mathbb{C}. \tag{84}$$

Actually, as shown in Chapter 10, the simultaneous spectral analysis of the Lax pair (73) implies that $q(x,t)$ can be expressed as follows: q is given by (74), where the sectionally analytic function μ, in addition to a jump across the real k-axis, also has a jump across the positive imaginary axis,

$$\mu = \begin{cases} \mu_1, & \arg k \in [0,\pi], \\ \mu_2, & \arg k \in [\pi, 3\pi/2], \\ \mu_3, & \arg k \in [3\pi/2, 2\pi], \end{cases}$$

where $(x,t) \in \Omega$ and

$$\mu_1 - \mu_3 = -e^{ikx - ik^2 t}\hat{q}_0(k), \quad k \in \mathbb{R}^+,$$
$$\mu_2 - \mu_1 = e^{ikx - ik^2 t}[\hat{q}_0(k) - \tilde{g}(k)], \quad k \in \mathbb{R}^-,$$
$$\mu_2 - \mu_3 = -e^{ikx - ik^2 t}\tilde{g}(k), \quad k \in i\mathbb{R}^-.$$

The solution of the above RH problem can be expressed in a closed form for all $(x,t) \in \Omega$ and for $k \notin \mathbb{R} \cup i\mathbb{R}^+$:

$$\mu = \frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{ilx - il^2 t}\hat{q}_0(l)\frac{dl}{l-k} - \frac{1}{2\pi i}\int_{\partial D^+} e^{ilx - il^2 t}\tilde{g}(l)\frac{dl}{l-k}.$$

Then (74) yields (83).

The spectral functions $\hat{q}_0(k)$ and $\tilde{g}(k)$ are directly related to the x-part and t-part, respectively, of the Lax pair (73): Let

$$\psi(x,k) \doteqdot \mu(x,0,k), \quad \varphi(t,k) \doteqdot \mu(0,t,k). \tag{85}$$

Then

$$\hat{q}_0(k) = -\psi(0,k); \left\{ \begin{array}{l} \psi_x(x,k) - ik\psi(x,k) = q_0(x), \quad 0 < x < \infty, \quad \operatorname{Im} k \leq 0, \\ \lim_{x\to\infty} e^{-ikx}\psi(x,k) = 0; \end{array} \right. \tag{86}$$

$$\tilde{g}(k) = e^{ik^2T}\varphi(T,k) \begin{cases} \varphi_t(t,k) + ik^2\varphi(t,k) = iq_x(0,t) - kq(0,t), & 0 < t < T, \quad k \in \mathbb{C}, \\ \varphi(0,k) = 0. \end{cases} \tag{87}$$

We emphasize that the conceptual steps needed for the solution of the NLS equation (81) in the domain Ω are *identical* to those used above for the solution of the linearized version of the NLS. The main technical difference is that the *scalar* RH problem for μ is now replaced by a 2×2 *matrix* RH for the sectionally analytical function M. Indeed, it will be shown in Chapter 16 that in the case of the defocusing NLS ((81) with $\lambda = -1$), $q(x,t)$ is given by

$$q(x,t) = 2i \lim_{k\to\infty} (kM(x,t,k))_{12}, \quad (x,t) \in \Omega,$$

where the sectionally analytic 2×2 matrix-valued function M as $k \to \infty$ has the asymptotic behavior (76b) and has "jumps" across both the real and imaginary axes of the complex k-plane. These jumps are determined by the spectral functions $(a(k), b(k))$ and $(A(k), B(k))$ which are the analogues of $\hat{q}_0(k)$ and $\tilde{g}(k)$. Actually, just as in the case for the linear problem, the spectral functions are directly related to the x-part and the t-part of the associated Lax pair, i.e., equations (78) (with $r = \bar{q}$): In analogy with (86) we now have the following equations (see [49]):

$$\begin{pmatrix} b(k) \\ a(k) \end{pmatrix} = \Psi(0,k), \quad \operatorname{Im} k \geq 0, \tag{88}$$

where for $0 < x < \infty$ and $\operatorname{Im} k \geq 0$, the vector $\Psi(x,k)$ satisfies the second column of the x-part of the Lax pair evaluated at $t = 0$,

$$\Psi_x(x,k) + 2ik \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Psi(x,k) = \begin{pmatrix} 0 & q_0(x) \\ \overline{q_0}(x) & 0 \end{pmatrix} \Psi(x,k). \tag{89}$$

Similarly, in analogy with (87) we now have the following equations:

$$\begin{pmatrix} B(k) \\ \overline{A(\bar{k})} \end{pmatrix} = \Phi(0,k), \quad k \in \mathbb{C}, \tag{90}$$

where for $0 < t < T$ and $k \in \mathbb{C}$ the vector $\Phi(t,k)$ satisfies the second column of the t-part of the Lax pair evaluated at $x = 0$,

$$\Phi_t(t,k) + 4ik^2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Phi(t,k) = \begin{pmatrix} -i|q(0,t)|^2 & -iq_x(0,t) + 2kq(0,t) \\ -i\bar{q}_x(0,t) + 2k\bar{q}(0,t) & i|q(0,t)|^2 \end{pmatrix} \Phi(t,k). \tag{91}$$

We note that the equations for Ψ and Φ, in contrast with the equations for ψ and ϕ, *cannot* be solved in a closed form. Hence, the spectral functions for the NLS, in contrast with the functions $\hat{q}_0(k)$ and $\tilde{g}(k)$, *cannot* be expressed in terms of explicit integrals but are expressed in terms of the solutions of (89) and (91) (which are equivalent to linear integral equations of the Volterra type). Similarly, the solution of the associated matrix RH problem for M *cannot* be solved in a closed form, but it is characterized through the solution of a linear integral equation of a Fredholm type. Hence, $q(x,t)$ cannot be written in an explicit form (like (83)), but it involves M:

$$q(x,t) = -\frac{1}{\pi}\left\{\int_{\partial D_3} \overline{\Gamma(\bar{k})} e^{-2ikx-4ik^2t} M_{11}^{+} dk \right. \\ \left. + \int_{-\infty}^{\infty} \gamma(k) e^{-2ikx-4ik^2t} M_{11}^{+} dk + \int_{0}^{\infty} |\gamma(k)|^2 M_{12}^{+} dk \right\}, \tag{92}$$

where ∂D_3 denotes the oriented boundary of the third quadrant of the complex k-plane, M_{11} and M_{12} are the (11) and (23) components of the solution M of the associated RH problem and

$$\gamma(k) = \frac{b(k)}{\overline{a(k)}}, \quad \Gamma(k) = \frac{1}{a(k)}\left[\frac{\overline{A(\bar{k})}}{\overline{B(\bar{k})}} a(k) - b(k)\right]^{-1}. \tag{93}$$

Equation (83) is the linear limit of (92). Indeed, for small q, we have $M_{11} \sim 1$, $M_{12} \sim 0$, $\gamma(k) \sim \hat{q}_0(k)$, $\overline{\Gamma(\bar{k})} \sim \tilde{g}(k)$, and (92) becomes (83) (after the transformation $-2k \to k$).

The situation for the modified KdV (mKdV) and KdV equations is similar (see [49], [50], [51], [52], [53]). Furthermore, these results can also be extended to integrable nonlinear PDEs formulated on the finite interval (see [54], [55]).

I.5.3. The Dirichlet to Neumann Map

If $q(x,t)$ satisfies the defocusing NLS, then $q(x,t)$ is given by the RHS of (92), where the functions $\gamma(k)$ and $\Gamma(k)$ are characterized through $q_0(x)$ and $\{q(0,t), q_x(0,t)\}$, respectively (see equations (93), (88)–(91)); furthermore M is the solution of a 2×2 matrix RH problem uniquely defined in terms of $\gamma(k)$ and $\Gamma(k)$. For a given initial-boundary value problem either $q(0,t)$ or $q_x(0,t)$ is an unknown function; therefore the last (and most difficult step) is the determination of $\Gamma(k)$ in terms of $q_0(x)$ and the given boundary data. This can be achieved through the analysis of the global relation, which for the defocusing NLS is given by

$$a(k)B(k) - b(k)A(k) = e^{4ik^2T} c^{+}(k), \quad \operatorname{Im} k \geq 0, \tag{94}$$

where $c^{+}(k)$ is a function analytic for $\operatorname{Im} k > 0$ and of $O(1/k)$ as $k \to \infty$. For q small, we have $a(k) \sim 1$, $A(k) \sim 1$, $b(k) \sim \hat{q}_0(k)$, $B(k) \sim \tilde{g}(k)$, and (94) becomes the associated global relation for (72) (see (44) evaluated at $t = T$)

$$\hat{q}_0(k) - i\int_0^T e^{ik^2s} q_x(0,s)ds + k\int_0^T e^{ik^2s} q(0,s)ds = e^{ik^2T} c^{+}(k), \quad \operatorname{Im} k \leq 0, \tag{95}$$

where $c^+(k)$ denotes the Fourier transform of $q(x, T)$ (the discrepancy between $\operatorname{Im} k \geq 0$ and $\operatorname{Im} k \leq 0$ is due to the fact that for the analysis of the NLS, k is replaced by $-2k$).

Let us consider the Dirichlet problem, i.e., $q(0, t) = g_0(t)$, where $g_0(t)$ is a given smooth function. The generalized transform method is based on the fact that the t-integral of $q_x(0, t)$ appearing in the integral representation for $q(x, t)$ *can* be eliminated using the algebraic manipulation of the equation obtained from (95) by the invariant transformation $k \to -k$. (This equation implies that $\tilde{g}(k)$ can be expressed in terms of $\hat{q}_0(-k)$ and of the t-transform of $g_0(t)$.) Unfortunately, the analogue of this approach works only for a very particular class of boundary conditions. Indeed, the spectral functions for the NLS involve the functions $\Psi(t, k)$ and $\Phi(t, k)$ and these functions do *not* remain invariant if k is replaced by $-k$. The class of boundary conditions for which the unknown spectral functions can be obtained through the algebraic manipulation of the global relation will be referred to as *linearizable* (see [50], [51], [56]). For the NLS, such a boundary condition is the homogeneous Robin boundary condition

$$q_x(0, t) - \chi q(0, t) = 0, \quad \chi > 0. \tag{96a}$$

In this case, $\Gamma(k)$ *can* be explicitly expressed in terms of χ and of $\{a(k), b(k)\}$: $\Gamma(k)$ is given by the second of equations (93) with

$$\frac{B(k)}{A(k)} = -\frac{2k + i\chi}{2k + i\chi}\frac{b(-k)}{a(-k)}. \tag{96b}$$

In the general case, it is *not* possible to solve the global relation for B/A, but it is necessary to solve the global relation for the unknown boundary value. It was shown earlier in section I.1.1 that the global relation for linear PDEs can be solved explicitly for the unknown boundary values. It is remarkable that in the nonlinear case the global relation can also be solved *explicitly* (see [57], [58], [59]). This result makes crucial use of the so-called Gel'fand–Levitan–Marchenko (GLM) representation [60]. For example, for the NLS with $q_0(x) = 0$ and $q(0, t) = g_0(t)$ the following formula is valid:

$$q_x(0, t) = g_0(t)M_2(t, t) - \frac{e^{-\frac{i\pi}{4}}}{\sqrt{\pi}} \int_0^t \frac{\partial M_1}{\partial \tau}(t, 2\tau - t)\frac{d\tau}{\sqrt{t - \tau}}, \quad 0 < t < T, \tag{97}$$

where $M_1(t, s)$ and $M_2(t, s)$ are the functions characterizing the GLM representation of the vector $\Phi(t, k)$ (see Chapter 18 for details). In the linear limit, we have $M_2(t, t) \sim 0$, $M_1(t, 2\tau - t) \sim g_0(\tau)$, and (97) yields the Dirichlet to Neumann map associated with the linearized NLS (compare with (23) for the analogous map for the heat equation),

$$q_x(0, t) = -\frac{e^{-\frac{i\pi}{4}}}{\sqrt{\pi}} \int_0^t \dot{g}_0(\tau)\frac{d\tau}{\sqrt{t - \tau}}. \tag{98}$$

There exist certain initial-boundary value problems, for which all relevant boundary values are prescribed as boundary conditions (see [61], [62], [63]). For these simpler problems, the simultaneous spectral analysis yields directly the effective representation of the solution and there is *no* need to consider the global relation. An example of such an initial-boundary value problems is discussed in Chapter 19.

One of the important advantages of the new method for integrable PDEs is that it can be used for the study of the asymptotic properties of the solution. The large t behavior of the NLS, sG (sine Gordon) and KdV equations on the half-line is studied in [64], [65], [66], respectively. The relevant analysis is based on the RH problem formulation and on the Deift–Zhou method (see [67], [68]). The latter method is an elegant nonlinearization of the steepest descent method, and it yields rigorous asymptotic results for RH problems with exponential (x, t) dependence. In our opinion this result is one of the most important developments in the theory of integrable systems in particular, and in the theory of RH problems in general, and thus it is quite satisfying that the new method gives rise to RH problems precisely of the type that can be analyzed by the Deift–Zhou method. Also, by employing an essential extension of the Deift–Zhou method introduced by Deift, Venakides, and Zhou (the so-called g-function mechanism) [69], [70], it is possible to calculate asymptotically fully nonlinear waveforms such as those in the zero-dispersion limit [71].

I.5.3.1. The Ehrenpreis Fundamental Principle

Finally, we note that the method presented here is deeply related to the so-called fundamental principle. Indeed, the expression of $q(x, y)$ for linear equations has *explicit exponential* (x, y) *dependence* consistent with the Euler–Palamodov–Ehrenpreis representation (see [72], [73], [74]). The expression of $q(x, t)$ for nonlinear equations involves an RH problem whose jump matrix has *an explicit exponential* (x, t) *dependence*. Thus, the new method provides the *concrete implementation as well as the nonlinearization* of this fundamental representation.

Regarding the Palamodov–Ehrenpreis representation, we note that in 1950 Schwartz posed the problem of whether, given a polynomial P on $\mathbb{C}^n$, an elementary solution of the differential operator $P(i\partial/\partial x)$, $x \in \mathbb{R}^n$, always exists, i.e., if there exists a distribution E solving $P(i\partial/\partial x)E = \delta$, where δ is the Dirac delta function. The existence of such an elementary solution was established independently by Malgrange and by Ehrenpreis using *nonconstructive* proofs. Thus in the same decade, techniques of functional analysis were used to try to construct this elementary solution explicitly. For example, if one considers the equation

$$P\left(i\frac{\partial}{\partial x}\right)q(x) = 0, \quad x \in \Omega, \tag{99}$$

where Ω is a convex domain in $\mathbb{R}^n$, it follows that $q(x) = e^{ik\cdot x}$, $k \in \mathbb{C}^n$, is a solution of (99) if $P(k) = 0$. For $n = 1$, the Euler principle states that every solution of (99) is a linear combination of exponentials. The generalization of Euler's principle for $n > 1$ was established by Ehrenpreis and by Palamodov. The statement of this result, called by Ehrenpreis the *fundamental principle*, is as follows: *If q is a solution of* (99) *in an appropriate functional space in a convex domain Ω in $\mathbb{R}^n$, then there exists a measure μ, whose support lies in $P^{-1}(0)$, such that*

$$q(x) = \int_{P^{-1}(0)} A(k, x)e^{-ik\cdot x}d\mu(k), \quad x \in \mathbb{R}^n, \quad k \in \mathbb{C}^n,$$

where A is a polynomial such that the trigonometric polynomial $A(k, x)e^{-ik\cdot x}$ is a solution of (99) *for every fixed $k \in P^{-1}(0)$.*

The proof of this beautiful result is based on an interpolation theorem for holomorphic functions with a given growth rate on a subdomain of $\mathbb{C}^n$; the measure μ is *not* constructed explicitly.

Recently, using certain generalized d-bar formulas derived by Henkin [74], there has been some progress in determining the explicit form of the measure $d\mu$ in the case of a smooth, bounded, convex domain; see [75].

An elementary implication of the Ehrenpreis principle is that for the equation (42) formulated in $0 < x < \infty$, $0 < t < T$, there exists a measure such that

$$q(x,t) = \int e^{ikx-\omega(k)t} d\mu(k).$$

The method introduced in this book expresses $d\mu(k)$ *explicitly* in terms of the given initial and boundary data.

Part I

A New Transform Method for Linear Evolution Equations

Chapter 1

Evolution Equations on the Half-Line

The application of the new method to linear evolution PDEs formulated either on the half-line or on the finite interval was discussed in section I.1 of the introduction. In this part of the book we will present the derivation of the results mentioned in the introduction and will also discuss some additional examples.

We will concentrate on the linear evolution PDE (42), i.e.,

$$q_t + w(-i\partial_x)q = 0, \tag{1.1}$$

where

$$w(k) \text{ is a polynomial of degree } n \text{ and } \operatorname{Re} w(k) \geq 0 \text{ for } k \text{ real.} \tag{1.2}$$

The above restriction on $w(k)$ ensures that the initial-value problem of (1.1) on the full line is well posed.

Let α_n denote the coefficient of k^n, i.e.,

$$w(k) = \alpha_n k^n + \alpha_{n-1}k^{n-1} + \cdots + \alpha_0, \quad \alpha_n \neq 0. \tag{1.3}$$

The large k limit of the condition $\operatorname{Re} w(k) > 0$ implies that if n is odd, then necessarily α_n is either i or $-i$, whereas if n is even, then $\operatorname{Re} \alpha_n \geq 0$.

The simplest way to determine the function $w(k)$ corresponding to a given linear evolution PDE is to substitute in the given PDE the following exponential which is an exact solution of (1.1):

$$e^{ikx-w(k)t}, \quad k \in \mathbb{C}. \tag{1.4}$$

We first introduce some useful notations.

- It will turn out that an initial-boundary value problem for (1.1) on the half-line requires N boundary conditions, where

$$N = \begin{cases} \frac{n}{2}, & n \quad \text{even}, \\ \frac{n+1}{2}, & n \quad \text{odd}, \alpha_n = i, \\ \frac{n-1}{2}, & n \quad \text{odd}, \alpha_n = -i, \end{cases} \tag{1.5}$$

  and α_n is the coefficient of k^n in $w(k)$; see (1.3).

- The given initial condition will be denoted by $q_0(x)$ and its Fourier transform by $\hat{q}_0(k)$,

$$q(x,0) = q_0(x), \quad 0 < x < \infty; \quad \hat{q}_0(k) = \int_0^\infty e^{-ikx} q_0(x)dx, \quad \operatorname{Im} k \geq 0. \tag{1.6}$$

  It is assumed that $q_0(x)$ has sufficient smoothness and sufficient decay as $x \to \infty$.

- The t-transforms of the boundary values will be denoted by $\tilde{g}_j(k)$,

$$\tilde{g}_j(k) = \int_0^T e^{ks} \partial_x^j q(0,s)ds, \quad k \in \mathbb{C}, \quad j = 0, 1, \ldots, n-1. \tag{1.7}$$

- If the function q or one of its derivatives is specified as a boundary condition, then this function will be denoted by $g_j(t)$ and its t-transform by $G_j(k)$,

$$\partial_x^j q(0,t) = g_j(t), \quad 0 < t < T; \quad G_j(k) = \int_0^T e^{ks} g_j(s)ds, \quad k \in \mathbb{C}. \tag{1.8}$$

  It is assumed that $g_j(t)$ has sufficient smoothness and that it is compatible with $q_0(x)$ at $x = t = 0$, i.e., $g_j(0) = (d/dx)^j q(0)$.

  If q and its first $N-1$ derivatives are specified as boundary conditions, i.e., if j in (1.8) takes the values $0, 1, \ldots, N-1$, then we will refer to this problem as the *canonical* problem.

- It will turn out that the integral representation which will be derived for q is also valid if T is replaced by t in (1.8); the relevant integral will be denoted by $G_j(k,t)$, i.e.,

$$G_j(k,t) = \int_0^t e^{ks} g_j(s)ds, \quad k \in \mathbb{C}, \quad 0 < t < T. \tag{1.9}$$

- $\mathbb{C}^+$ and $\mathbb{C}^-$ will denote the upper half ($\operatorname{Im} k > 0$) and the lower half ($\operatorname{Im} k < 0$) of the complex k-plane. The domain D is defined by

$$D = \{k \in \mathbb{C}, \quad \operatorname{Re} w(k) < 0\}. \tag{1.10}$$

  D^+ and D^- will denote the part of D in $\mathbb{C}^+$ and $\mathbb{C}^-$,

$$D^+ = D \cap \mathbb{C}^+, \quad D^- = D \cap \mathbb{C}^-. \tag{1.11}$$

- The asymptotic form of D, D^+, D^- as $k \to \infty$ will be denoted by D_R, D_R^+, D_R^-, respectively, i.e.,

$$D_R = \{k \in D, \quad |w(k)| > R, \quad R \text{ large}\}, \tag{1.12}$$

$$D_R^+ = D_R \cap \mathbb{C}^+, \quad D_R^- = D_R \cap \mathbb{C}^-. \tag{1.13}$$

  Outside the curve defined by $|w(k)| = R$, $R > 0$ and sufficiently large, ∂D is the union of smooth disjoint simple contours which approach asymptotically, as $k \to \infty$, the rays defined by

$$\operatorname{Re}\left(k + \frac{\alpha_{n-1}}{n\alpha_n}\right)^n = 0,$$

  where α_n and α_{n-1} are the coefficients of k^n and k^{n-1} in $w(k)$; see (1.3).

Proposition 1.1 (a general integral representation). Let $q(x, t)$ satisfy the linear evolution PDE (1.1) in the domain

$$\Omega = \{0 < x < \infty,\ 0 < t < T\}, \tag{1.14}$$

where $w(k)$ satisfies the restrictions specified in (1.2). Define the polynomials $c_j(k)$, $j = 0, \dots, n-1$, by the identity

$$\sum_{j=0}^{n-1} c_j(k)\partial_x^j = i\left.\frac{w(k)-w(l)}{k-l}\right|_{l=-i\partial_x}. \tag{1.15}$$

Assume that $q(x, t)$ is a sufficiently smooth (up to the boundary of Ω) solution of (1.1), which has sufficient decay, as $x \to \infty$, uniformly in $0 \le t \le T$. Then, $q(x, t)$ is given by

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-w(k)t}\hat{q}_0(k)dk - \frac{1}{2\pi}\int_{\partial D^+} e^{ikx-w(k)t}\tilde{g}(k)dk, \tag{1.16}$$

where ∂D^+ is the oriented boundary of the domain D^+ defined in (1.11) with the orientation that D^+ is to the left of ∂D^+, $\hat{q}_0(k)$ is defined by (1.6), and the function $\tilde{g}(k)$ is defined in terms of the t-transforms of the boundary values $\{\partial_x^j q(0, t)\}_0^{n-1}$ (see (1.7)) by the formula

$$\tilde{g}(k) = \sum_{j=0}^{n-1} c_j(k)\tilde{g}_j(w(k)), \quad k \in \mathbb{C}. \tag{1.17}$$

Furthermore, the following global relation is valid:

$$\hat{q}_0(k) - \tilde{g}(k) = e^{w(k)T}\int_0^{\infty} e^{-ikx}q(x,T)dx, \quad \operatorname{Im} k \le 0. \tag{1.18}$$

Equations (1.16) and (1.18) are also valid if T in these equations is replaced by t, in which case these equations become (45) and (44), respectively, of the introduction.

Proof. A particular solution of the formal adjoint of (1.1) is $\tilde{q} = \exp[-ikx + w(k)t]$: thus we start with (43) of the introduction, where $\{c_j\}_1^{n-1}$ are to be determined. Simplifying (43) and replacing q_t by $-w(-i\partial_x)q$ we find

$$\frac{w(k) - w(-i\partial_x)}{-ik + \partial_x}q = \sum_{j=0}^{n-1} c_j(k)\partial_x^j q,$$

which is (1.15).

Equation (43) and the application of Green's theorem in the domain Ω and in the domain $\{0 < x < \infty, 0 < s < t\}$, yield (1.18) and (44), respectively. Solving the latter equation for $\hat{q}(k, t)$ and then using the inverse Fourier transform, we find an equation similar to (45), where the second integral in the RHS involves an integral along the real axis instead of an integral along ∂D^+. The fact that the real axis can be deformed to

∂D^+ is a consequence of the following: (a) The function $\text{Re}\, w(k)$, which characterizes the domain D, is a harmonic function of (k_R, k_I), and thus the set $\mathbb{C}\backslash\partial D$ is the union of disjoint unbounded simply connected open sets. (b) The second integral in the RHS of (45) involves the integrand

$$e^{ikx}\int_0^t e^{-w(k)(t-s)}\sum_{j=0}^{n-1} c_j(k)\partial_x^j q(0,s)ds, \tag{1.19}$$

and this expression is bounded and analytic in $E^+ = \mathbb{C}^+\backslash D^+$ and of order e^{ikx}/k as $k\to\infty$ (the exponential $\exp[-w(k)(t-s)]$ is bounded for $\text{Re}\, w(k) \geq 0$ and integration by parts implies that as $k\to\infty$ the expression in (1.19) is of order $\exp[ikx]c_{n-1}(k)/w(k))$. Hence, Jordan's lemma in the domain E^+ implies that the real axis can be deformed up to the boundary of E^+ which coincides with ∂D^+.

Replacing t by T in (45) is equivalent to adding a term involving an integrand similar to the expression in (1.19) where the integral from 0 to t is now replaced by the integral from t to T. However, in this case $\exp[-w(k)(t-s)]$ is bounded and analytic in D^+, and thus Jordan's lemma in D^+ implies that the contribution of this term vanishes. □

Remark 1.1. For T finite, $\tilde{g}(k)$ is an entire function, thus ∂D^+ can be replaced with *any* other contour in $\mathbb{C}^+$ which as $k\to\infty$ approaches ∂D^+. For $T=\infty$, ∂D^+ can be replaced with ∂D_R^+.

Remark 1.2. For the rigorous analysis of well-posedness as well as for certain other applications it is convenient to use the representation (1.16) instead of (45). On the other hand, (45) has the advantage that it is consistent with causality. In what follows we will use either (1.16) or (45) depending on the circumstances. Similarly, we will use the global relation either in the form (1.18) or in the form (44).

Example 1.1 (the heat equation). Substituting the expression (1.4) in the heat equation

$$q_t = q_{xx} \tag{1.20a}$$

we find

$$w(k) = k^2. \tag{1.20b}$$

The domain D is defined by $\text{Re}\,(k^2) < 0$, i.e., $\cos[2\arg k] < 0$. Hence,

$$2\arg k \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) + 2m\pi, \quad m = 0, 1.$$

Thus

$$D = \left\{\arg k \in \left(\frac{\pi}{4}, \frac{3\pi}{4}\right) \cup \left(\frac{5\pi}{4}, \frac{7\pi}{4}\right)\right\}. \tag{1.20c}$$

Equation (1.15) implies

$$\sum_{j=0}^{1} c_j(k)\partial_x^j = i\left.\frac{k^2-l^2}{k-l}\right|_{l=-i\partial_x} = ik + \partial_x.$$

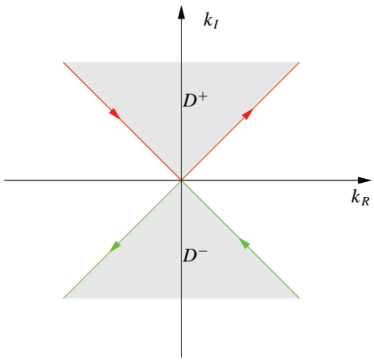


Figure 1.1. *The domains D^+ and D^- for the heat equation.*

Thus

$$\tilde{g}(k) = ik\tilde{g}_0(k^2) + \tilde{g}_1(k^2). \tag{1.20d}$$

Hence the solution of the heat equation (1.20a) is given by (1.16) with $w = k^2$, $\tilde{g}$ defined by (1.20d), and ∂D^+ depicted in red in Figure 1.1 (the green contour will be needed for the finite interval).

Example 1.2 (a PDE with a second order derivative). Let q satisfy the PDE

$$q_t = q_{xx} + \beta q_x, \quad \beta > 0. \tag{1.21a}$$

This PDE appears in several applications including pharmacokinetics [76]. Furthermore, solving (1.21a) on the half-line is equivalent to solving the heat equation in a linearly moving domain. Indeed, let $u(\xi, \tau)$ satisfy the heat equation in the domain

$$\tilde{\Omega}(\tau) = \{\beta\tau < \xi < \infty, \quad \tau > 0\}.$$

Let

$$t = \tau, \quad x = \xi - \beta\tau, \quad q(x,t) = u(\xi, \tau).$$

Then

$$\partial_\tau = \partial_t - \beta\partial_x, \quad \partial_\xi = \partial_x$$

and the heat equation in $\tilde{\Omega}(\tau)$ becomes (1.21a) in Ω.

Substituting the expression (1.4) in (1.21a) we find

$$w(k) = k^2 - i\beta k. \tag{1.21b}$$

Letting $k = k_R + ik_I$ in $\operatorname{Re} w(k) < 0$ we find

$$D = \left\{k_R^2 - k_I^2 + \beta k_I < 0\right\}. \tag{1.21c}$$

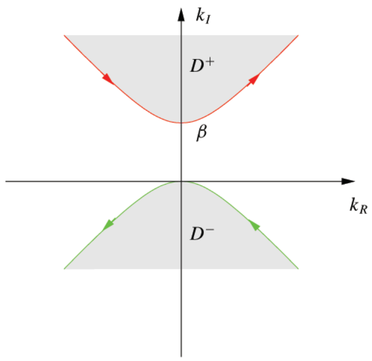


Figure 1.2. *The domains D^+ and D^- for* (1.21a).

Equation (1.15) implies

$$\sum_{j=0}^{1} c_j(k)\partial_x^j = i\,\frac{(k^2-l^2)-i\beta(k-l)}{k-l} = i(k-i\partial_x)+\beta.$$

Thus

$$\tilde{g}(k) = (ik+\beta)\tilde{g}_0(w(k)) + \tilde{g}_1(w(k)). \tag{1.21d}$$

Hence the solution of (1.21a) on the half-line is given by (1.16) with w, $\tilde{g}$ defined by (1.21b), (1.21d), respectively, and ∂D^+ depicted in red in Figure 1.2.

Example 1.3 (a PDE with a third order derivative). Let q satisfy the PDE

$$q_t + q_{xxx} = 0. \tag{1.22a}$$

In this case

$$w(k) = -ik^3. \tag{1.22b}$$

Using

$$w(k) = -i|k|^3\left[\cos(3\arg k) + i\sin(3\arg k)\right],$$

it follows that D is defined by $\sin(3\arg k) < 0$, i.e.,

$$3\arg k \in (\pi, 2\pi) + 2m\pi, \quad m = 0, 1, 2.$$

Thus

$$D = \left\{\arg k \in \left(\frac{\pi}{3}, \frac{2\pi}{3}\right) \cup \left(\pi, \frac{4\pi}{3}\right) \cup \left(\frac{5\pi}{3}, 2\pi)\right)\right\}. \tag{1.22c}$$

Equation (1.15) implies

$$\sum_{j=0}^{2} c_j(k)\partial_x^j = i(-i)\left.\frac{k^3-l^3}{k-l}\right|_{l=-i\partial_x} = k^2 + (-i\partial_x)^2 + k(-i\partial_x).$$

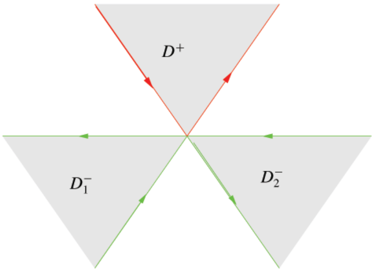


Figure 1.3. *The domains D^+ and D^- for* (1.22a).

Thus

$$\tilde{g} = k^2\tilde{g}_0(w(k)) - ik\tilde{g}_1(w(k)) - \tilde{g}_2(w(k)). \tag{1.22d}$$

The domains D^+ and D^- are shown in Figure 1.3.

Example 1.4 (another PDE with a third order derivative). Let q satisfy the PDE

$$q_t - q_{xxx} = 0. \tag{1.23a}$$

In this case

$$w(k) = ik^3 \tag{1.23b}$$

and

$$D = \left\{\arg k \in \left(0, \frac{\pi}{3}\right) \cup \left(\frac{2\pi}{3}, \pi\right) \cup \left(\frac{4\pi}{3}, \frac{5\pi}{3}\right)\right\}, \tag{1.23c}$$

$$\tilde{g}(k) = -k^2\tilde{g}_0(w(k)) + ik\tilde{g}_1(w(k)) + \tilde{g}_2(w(k)). \tag{1.23d}$$

The domains D^+ and D^- are shown in Figure 1.4.

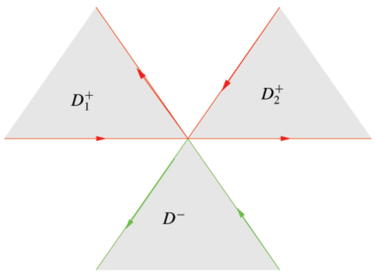


Figure 1.4. *The domains D^+ and D^- for* (1.23a).

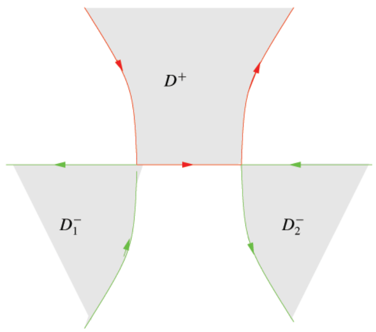


Figure 1.5. *The domains D^+ and D^- for the first Stokes equation.*

Example 1.5 (the first Stokes equation). Let q satisfy the first Stokes equation

$$q_t + q_{xxx} + q_x = 0. \tag{1.24a}$$

This is the linearized version of the celebrated Korteweg-de Vries (KdV) equation (1). In the case of water waves, the latter equation is the normalized form of the equation

$$\frac{\partial \eta}{\partial \tau} + \frac{3}{2}\sqrt{\frac{g}{h}}\frac{\partial}{\partial \xi}\left(\frac{1}{3}\lambda\frac{\partial^2 \eta}{\partial \xi^2} + \eta + \frac{1}{2}\eta^2\right) = 0, \quad \lambda = \frac{1}{8}h^3 - \frac{\sigma h}{\rho g},$$

where η is the elevation of the water above the equilibrium height h, σ is the surface tension, ρ is the density of the water, and g is the acceleration constant. This equation is the small amplitude, long wave limit of the equations describing inviscid, irrotational water waves.

The KdV equation usually appears without the q_x term because this equation is usually studied on the full line and then q_x can be eliminated using a Galilean transformation. However, for the half-line this transformation would change the domain to a wedge.

For (1.24a),

$$w(k) = ik - ik^3, \tag{1.24b}$$

$$D = \left\{k_I(3k_R^2 - k_I^2 - 1) < 0\right\}, \tag{1.24c}$$

and

$$\tilde{g}(k) = (k^2 - 1)\tilde{g}_0(w(k)) - ik\tilde{g}_1(w(k)) - \tilde{g}_2(w(k)). \tag{1.24d}$$

The domains D^+ and D^- are shown in Figure 1.5.

Example 1.6 (the second Stokes equation). Let q satisfy the second Stokes equation

$$q_t - q_{xxx} + q_x = 0. \tag{1.25a}$$

In the case of water waves this corresponds to dominant surface tension, $\sigma > g\rho h^2/8$.

In this case

$$w(k) = ik + ik^3, \tag{1.25b}$$

$$D = \left\{k_I(k_I^2 - 3k_R^2 - 1) < 0\right\}, \tag{1.25c}$$

and

$$\tilde{g}(k) = -(1+k^2)\tilde{g}_0(w(k)) + ik\tilde{g}_1(w(k)) + \tilde{g}_2(w(k)). \tag{1.25d}$$

The domains D^+ and D^- are shown in Figure 1.6.

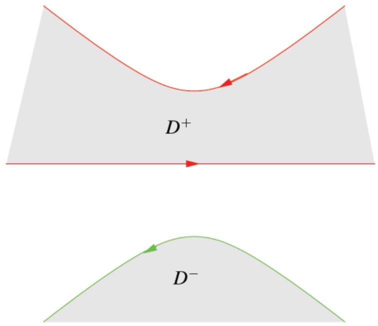


Figure 1.6. *The domains D^+ and D^- for the second Stokes equation.*

Example 1.7 (a PDE with a fifth order derivative). Let q satisfy the linear PDE

$$q_t + q_x - \partial_x^5 q = 0. \tag{1.26a}$$

In this case

$$w(k) = ik - ik^5, \tag{1.26b}$$

$$D = \left\{k_I(5k_R^4 + k_I^4 - 6k_R^2k_I^2 - 1) < 0\right\}, \tag{1.26c}$$

and

$$\tilde{g}(k) = (k^4-1)\tilde{g}_0(w(k)) - ik^3\tilde{g}_1(w(k)) - k^2\tilde{g}_2(w(k)) + ik\tilde{g}_3(w(k)) + \tilde{g}_4(w(k)). \tag{1.26d}$$

The domains D and D_R are shown in Figures 1.7(a) and 1.7(b), respectively.

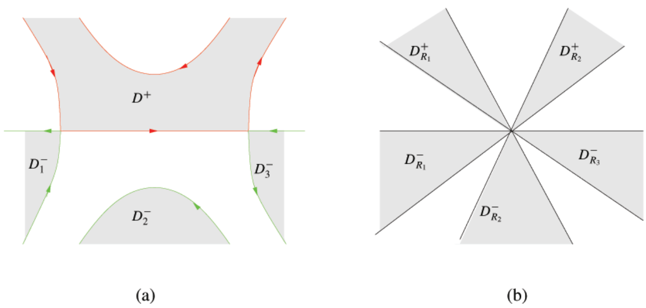


Figure 1.7. (a) *The domains D^+, D^-.* (b) *The domains D_R^+, D_R^-.*

The expression for $\tilde{g}(k)$ involves the t-transforms of all boundary values $\{\partial_x^j q(0,t)\}_0^{n-1}$. However, $n - N$ of these boundary values *cannot* be prescribed as boundary conditions;

thus (1.16) does *not* provide the effective solution of any initial-boundary value problem. However, using (1.16) *together* with the global relation (1.18), it is possible to obtain the solution of any boundary value problem for which N linearly independent combinations with constant coefficients of a subset of the set $\{\partial_x^j q(0,t)\}_0^{n-1}$ are prescribed as boundary conditions. For concreteness, in what follows we state the result for the case that the function q and its first $N-1$ derivatives are prescribed as boundary conditions (the canonical problem). Other types of boundary conditions, including Robin type, will be discussed in the examples.

Proposition 1.2 (the integral representation of the canonical problem). Define the integer N by (1.5). Let q satisfy (1.1) in the domain Ω defined by (1.14) and let q satisfy the following initial and boundary conditions:

$$q(x,0) = q_0(x), \quad 0 < x < \infty, \tag{1.27}$$

$$\partial_x^j q(0,t) = g_j(t), \quad 0 < t < T, \quad j = 0, 1, \dots, N-1. \tag{1.28}$$

Then $q(x,t)$ is given by

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-w(k)t}\hat{q}_0(k)dk - \frac{1}{2\pi}\int_{\partial D_R^+} e^{ikx-w(k)t}\tilde{g}(k)dk, \quad (x,t) \in \Omega, \tag{1.29}$$

where ∂D_R^+ is the oriented boundary of the domain D_R^+ defined in (1.13), $\hat{q}_0$ is defined in (1.6), and $\tilde{g}(k)$ is defined in terms of $\tilde{g}_j(k)$ by (1.17), where

$$\tilde{g}_j(k) = G_j(k), \quad j = 0, 1, \dots, N-1, \tag{1.30}$$

and the functions $\{\tilde{g}_j\}_N^{n-1}$ are determined as follows: The domain D_R^+ consists of N sectors, $\{D_{R,m}^+\}_1^N$, and in each of these sectors, say $D_{R,m}^+$, m fixed, $\{\tilde{g}_j(w(k))\}_N^{n-1}$ satisfy $n-N$ linear algebraic equations. These equations can be obtained from the global relation (1.18) by dropping the RHS and by replacing k with $\{\nu_{l,m}(k)\}_{l=1}^{n-N}$, where these $n-N$ functions are determined by the requirements that they leave $w(k)$ invariant and that they map $D_{R,m}^+$ to D_R^-, i.e., for fixed m,

$$w(k) = w(\nu_{l,m}(k)), \;\; k \in D_{R,m}^+, \quad \nu_{l,m}(k) \in D_R^-, \;\; l = 1, \dots, n-N. \tag{1.31}$$

Proof. It was noted in the proof of Proposition 1.1 that D consists of the union of disjoint unbounded simply connected open sets. For k large, $w(k) \sim \alpha_n k^n$ and D approaches D_R which consists of the following n sectors:

$$\arg\alpha_n + n\arg k \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) + 2m\pi, \;\; m = 0, \dots, n-1. \tag{1.32}$$

If n is odd and $\alpha_n = -i$ (i.e., $\arg\alpha_n = -\pi/2$), then $(n-1)/2$ of these sectors are in $\mathbb{C}^+$, whereas if $\alpha_n = i$, then $(n+1)/2$ of these sectors are in $\mathbb{C}^+$. If n is even, using the restriction that $\arg\alpha_n \in [-\pi/2, \pi/2]$, it follows that $n/2$ sectors are in $\mathbb{C}^+$.

Using Jordan's lemma in D^+ we can replace ∂D^+ by ∂D_R^+.

For large k, the solutions of the equation $w(k) = w(\nu(k))$ become

$$\nu_m(k) \sim e^{\frac{2i\pi m}{n}} k, \quad m = 1, \ldots, n-1. \tag{1.33}$$

Thus, for each fixed sector $D_{R,m}^+$ there exist $n-N$ functions $\{\nu_{l,m}\}_1^{n-N}$ which map this sector to the $n-N$ sectors of D_R^-. Hence, replacing k by $\nu_{l,m}$ in the global relation (1.18) we find

$$\hat{q}_0(\nu_{l,m}) - \sum_{j=0}^{n-1} c_j(\nu_{l,m})\tilde{g}_j(w(k)) = e^{w(k)T}\hat{q}_T(\nu_{l,m}), \quad l = 1, \ldots, n-N, \tag{1.34}$$

where $\hat{q}_T(k)$ denotes the Fourier transform of $q(x,T)$. The first N terms of the summation in the LHS of (1.34) are the known functions $G_j(w(k))$, and thus (1.34) can be considered as $n-N$ equations for the $n-N$ unknown functions $\{\tilde{g}_j\}_N^{n-1}$. The solution of this system yields for each $\tilde{g}_j$, $j = N, \ldots, n-1$, an expression corresponding to the homogeneous version of (1.34) plus an expression corresponding to the functions $\exp[w(k)T]\hat{q}_T(\nu_{l,m})$. However, the contribution to the solution $q(x,t)$ of the latter part vanishes. Indeed, the determinant of the relevant system is independent of k (see [77]), and hence the nonhomogeneous part of (1.34) yields an integrand which consists of linear combinations (polynomials in k of order less than $n-1$) of the terms $\exp[w(k)(T-t)]\hat{q}_T(\nu_{l,m})$. But $T-t > 0$, thus $\exp[w(k)(T-t)]$ is bounded and analytic in D_R^+, whereas $\hat{q}_T(\nu_{l,m})$ is bounded and analytic in $D_{R,m}^+$ (since $\hat{q}_T(k)$ is well defined in $\mathbb{C}^-$), thus Jordan's lemma in $D_{R,m}^+$ yields the desired result. □

Remark 1.3. Suppose that N linear combinations with *constant* coefficients of a subset of the boundary values $\{\partial_x^j q(0,t)\}_0^{n-1}$ are prescribed as boundary conditions. Then, provided that a certain determinant does not vanish identically, it is still possible to construct $q(x,t)$: The determination of $\{\tilde{g}_j\}_0^{n-1}$ still involves (1.34); however, the relevant determinant is now a polynomial in k instead of a constant. Thus, if this polynomial has zeros in D_R^+, then the contour ∂D_R^+ must be deformed to avoid these zeros, or alternatively the contribution of these zeros can be computed via a residue calculation. This will be illustrated in the examples below.

Example 1.8 (the heat equation). Using the expression for $\tilde{g}(k)$ defined by (1.20d), the global relation (1.18) becomes

$$\hat{q}_0(k) - [ik\tilde{g}_0(k^2) + \tilde{g}_1(k^2)] = e^{k^2T}\hat{q}_T(k), \quad \operatorname{Im} k \le 0. \tag{1.35a}$$

The equation $k^2 = \nu^2$ has only one solution other than k, $\nu = -k$. Replacing k with $-k$ in (1.35a) we find

$$\hat{q}_0(-k) - \left[-ik\tilde{g}_0(k^2) + \tilde{g}_1(k^2)\right] = e^{k^2T}\hat{q}_T(-k), \quad \operatorname{Im} k \ge 0. \tag{1.35b}$$

(i) *The Dirichlet Problem.* The solution $q(x,t)$ is given by

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^2t}\hat{q}_0(k) - \frac{1}{2\pi}\int_{\partial D^+} e^{ikx-k^2t}\left[\hat{q}_0(-k) + 2ikG_0(k^2)\right]dk, \tag{1.36}$$

where ∂D^+ is as depicted in Figure 1.1. Furthermore $G_0(k^2)$ can be replaced by $G_0(k^2, t)$ (defined in (1.9)), in which case (1.36) becomes (16).

Indeed, in this case $g_0 = G_0$, and thus by solving the homogeneous version of (1.35b) for $\tilde{g}_1$ and inserting the resulting expression in $\tilde{g}$ we find

$$\tilde{g}(k) = 2ikg_0(k^2) + \hat{q}_0(-k), \quad \operatorname{Im} k \geq 0.$$

Hence (1.16) becomes (16).

(ii) *The Neumann Problem.* The solution $q(x, t)$ is given by

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^2t}\hat{q}_0(k) - \frac{1}{2\pi}\int_{\partial D^+} e^{ikx-k^2t}\left[2G_1(k^2) - \hat{q}_0(-k)\right]dk, \tag{1.37}$$

where $G_1(k)$ is the t-transform of the Neumann data (see (1.8)) and ∂D^+ is as shown in Figure 1.1. Furthermore, $G_1(k^2)$ can be replaced by $G_1(k^2, t)$ defined in (1.9).

Indeed, in this case $\tilde{g}_1 = G_1$, and thus by solving (1.35b) for $ik\tilde{g}_0(k^2)$ and inserting the resulting expression in $\tilde{g}$ we find

$$\tilde{g}(k) = 2G_1(k^2) - \hat{q}_0(-k) + e^{k^2T}\hat{q}_T(-k), \quad \operatorname{Im} k \geq 0.$$

Inserting this expression in (1.16) and noting that the contribution of the term $\exp[k^2T]\hat{q}_T(-k)$ vanishes, we find (1.37).

(iii) *The Robin Problem.* Let

$$q_x(0,t) - \gamma q(0,t) = g_R(t), \quad 0 < t < T, \quad \gamma \text{ real constant.} \tag{1.38a}$$

The solution $q(x, t)$ is given by

$$\begin{aligned} q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^2t}\hat{q}_0(k) - \frac{1}{2\pi}\int_{\partial D^+} e^{ikx-k^2t}\left[\frac{2k}{k+i\gamma}G_R(k^2)\right. \\ \left.-\frac{k-i\gamma}{k+i\gamma}\hat{q}_0(-k)\right] + H(-\gamma)2\gamma e^{\gamma x+\gamma^2 t}\left[G_R(-\gamma^2) - \hat{q}_0(i\gamma)\right], \end{aligned} \tag{1.38b}$$

where $H(\gamma)$ denotes the usual Heaviside function, ∂D^+ is as depicted in Figure 1.1, and $G_R(k)$ denotes the t-transform of the function g_R, i.e.,

$$G_R(k) = \int_0^T e^{ks}g_R(s)ds, \quad k \in \mathbb{C}. \tag{1.38c}$$

Furthermore, $G_R(k^2)$ and $G_R(-\gamma^2)$ can be replaced by $G_R(k^2, t)$ and $G_R(-\gamma^2, t)$ defined in (1.9).

Indeed, taking the t-transform of the boundary condition (1.38a) we find

$$\tilde{g}_1(k^2) = \gamma\tilde{g}_0(k^2) + G_R(k^2), \quad k \in \mathbb{C}. \tag{1.39}$$

Replacing $\tilde{g}_1$ by the RHS of (1.39) in the global relation (1.35b) we find

$$(ik-\gamma)\tilde{g}_0(k^2) + \hat{q}_0(-k) - G_R(k^2) = e^{k^2T}\hat{q}_T(-k), \quad \operatorname{Im} k \geq 0. \tag{1.40}$$

Solving this equation for $\tilde{g}_0$ and then inserting the resulting expression for $\tilde{g}_0$, as well as the expression for $\tilde{g}_1$ (equation (1.39)), in the definition of $\tilde{g}$ (equation (1.20d)) we find

$$\tilde{g}(k) = \frac{2k}{k+i\gamma}G_R(k^2) - \frac{k-i\gamma}{k+i\gamma}\hat{q}_0(-k) + \frac{k-i\gamma}{k+i\gamma}e^{k^2T}\hat{q}_T(-k), \quad \mathrm{Im}\, k \geq 0. \tag{1.41}$$

If $\gamma > 0$, $k+i\gamma \neq 0$ for $k \in D^+$, thus the last term in the RHS of (1.41) does *not* contribute to the solution $q(x,t)$. However, if $\gamma < 0$, this term yields the following contribution:

$$-\frac{1}{2\pi}\int_{\partial D^+} e^{ikx+k^2(T-t)}\frac{k-i\gamma}{k+i\gamma}\hat{q}_T(-k)dk$$
$$= -\frac{1}{2\pi}(2i\pi)e^{\gamma x-\gamma^2(T-t)}(-2i\gamma)\hat{q}_T(i\gamma) = -2\gamma e^{\gamma x-\gamma^2(T-t)}\hat{q}_T(i\gamma).$$

Evaluating (1.40) at $k = -i\gamma$ we find

$$e^{-\gamma^2T}\hat{q}_T(i\gamma) = \hat{q}_0(i\gamma) - G_R(-\gamma^2)$$

and (1.38b) follows.

Example 1.9 (a PDE with a second order derivative). For (1.21a), using

$$w(k) - w(\nu) = (k^2 - i\beta k) - (\nu^2 - i\beta\nu) = (k-\nu)[k+\nu-i\beta],$$

it follows that $\nu(k) = -k + i\beta$. Thus we consider the global relation (1.18) with $\tilde{g}(k)$ given by (1.21d) and we replace k with $\nu(k)$,

$$\hat{q}_0(-k+i\beta) - [-ik\tilde{g}_0(w(k)) + \tilde{g}_1(w(k))] = e^{w(k)T}\hat{q}_T(-k+i\beta), \quad k \in D^+, \tag{1.42}$$

where $w(k)$ is defined by (1.21b).

(i) *The Dirichlet Problem.* The solution $q(x,t)$ is given by (1.16) where $w(k) = k^2 - i\beta k$, ∂D^+ is as depicted in Figure 1.2, and $\tilde{g}(k)$ is given as

$$\tilde{g}(k) = \hat{q}_0(-k+i\beta) + (2ik+\beta)G_0(w(k)). \tag{1.43}$$

Furthermore, $G_0(k^2)$ can be replaced by $G_0(k^2, t)$.

Indeed, in this case $\tilde{g}_0 = G_0$, and thus by solving the homogeneous version of (1.42) for $\tilde{g}_1$ and then substituting the resulting expression in the definition of $\tilde{g}(k)$ (equation (1.21d)) we find (1.43).

(ii) *The Neumann Problem.* In this case

$$\tilde{g}(k) = \frac{1}{ik}\left[-(ik+\beta)\hat{q}_0(-k+i\beta) + (2ik+\beta)G_1(w(k))\right], \quad k \in D^+. \tag{1.44}$$

Indeed, solving (1.42) for $\tilde{g}_0$ and then substituting the resulting expression in the definition of $\tilde{g}(k)$ we find the RHS of (1.44) plus the term

$$\frac{(ik+\beta)}{ik}e^{w(k)T}\hat{q}_T(-k+i\beta).$$

The contribution of this term vanishes since $k = 0$ is outside D^+.

Example 1.10 (a PDE with a third order derivative). For (1.22a), $\alpha_n = -i$, thus $N = 1$. Also

$$\nu_1 = \alpha k, \quad \nu_2 = \alpha^2 k, \quad \alpha = e^{\frac{2i\pi}{3}}. \tag{1.45}$$

If $k \in D^+$, then $\nu_1 \in D_1^-$ and $\nu_2 \in D_2^-$; see Figure 1.3. Thus we consider the global relation (1.18) with $\tilde{g}$ given by (1.22d) and we replace k with ν_1 and with ν_2,

$$\hat{q}_0(\alpha k) - \alpha^2 k^2 \tilde{g}_0 + i\alpha k \tilde{g}_1 + \tilde{g}_2 = e^{-ik^3T}\hat{q}_T(\alpha k), \tag{1.46a}$$

$$\hat{q}_0(\alpha^2 k) - \alpha k^2 \tilde{g}_0 + i\alpha^2 k \tilde{g}_1 + \tilde{g}_2 = e^{-ik^3T}\hat{q}_T(\alpha^2 k), \quad k \in D^+. \tag{1.46b}$$

The Dirichlet Problem. The solution $q(x,t)$ is given by (1.16) where $w(k) = -ik^3$, ∂D^+ is as depicted in Figure 1.3, and $\tilde{g}(k)$ is given by

$$\tilde{g}(k) = 3k^2 G_0(-ik^3) - \alpha\hat{q}_0(\alpha k) - \alpha^2\hat{q}_0(\alpha^2 k), \quad \alpha = e^{\frac{2i\pi}{3}}, \quad k \in D^+. \tag{1.47}$$

Furthermore, $G_0(-ik^3)$ can be replaced by $G_0(-ik^3, t)$ defined in (1.9).

Indeed, we must solve the homogeneous version of (1.46) for $\tilde{g}_1$ and $\tilde{g}_2$ and then substitute the resulting expressions in the expression for $\tilde{g}$. Actually, instead of following this procedure it is more convenient to supplement (1.46) with the equation for $\tilde{g}$ and to eliminate $\tilde{g}_1$ and $\tilde{g}_2$ from these three equations: the equation for $\tilde{g}$ is

$$-k^2 G_0 + ik\tilde{g}_1 + \tilde{g}_2 = -\tilde{g}(k).$$

Multiplying the homogeneous version of (1.46a) by α, the homogeneous version of (1.46b) by α^2, adding the resulting equations to the equation for $\tilde{g}(k)$ above, and using the fact that

$$1 + \alpha + \alpha^2 = 0, \tag{1.48}$$

we find the expression for $\tilde{g}(k)$ given by the RHS of (1.47).

Example 1.11 (another PDE with a third order derivative). For (1.23a), $\alpha_n = i$, and thus $N = 2$. The functions ν_1 and ν_2 are still defined by (1.45). If $k \in D_1^+$, then $\nu_1 \in D^-$ and if $k \in D_2^+$, then $\nu_2 \in D^-$; see Figure 1.4. Thus in order to determine $\tilde{g}(k)$ for $k \in D_1^+$ and $k \in D_2^+$ we must use ν_1 and ν_2, respectively. Hence, we consider the global relation (1.18) with $\tilde{g}$ given by (1.23d), i.e.,

$$-k^2\tilde{g}_0 + ik\tilde{g}_1 + \tilde{g}_2 = \tilde{g}(k)$$

and replace k with either ν_1 or ν_2:

$$\hat{q}_0(\alpha k) + \alpha^2 k^2 \tilde{g}_0 - i\alpha k \tilde{g}_1 - \tilde{g}_2 = e^{ik^3T}\hat{q}_T(\alpha k), \quad k \in D_1^+, \tag{1.49a}$$

$$\hat{q}_0(\alpha^2 k) + \alpha k^2 \tilde{g}_0 - i\alpha^2 k \tilde{g}_1 - \tilde{g}_2 = e^{ik^3T}\hat{q}_T(\alpha^2 k), \quad k \in D_2^+. \tag{1.49b}$$

The Canonical Problem. In this case

$$q(0,t) = g_0(t), \quad q_x(0,t) = g_1(t). \tag{1.50}$$

The solution $q(x,t)$ is given by (1.16) where $w(k) = ik^3$, ∂D^+ is as depicted in Figure 1.4, and $\tilde{g}(k)$ is given by

$$\tilde{g}(k) = \begin{cases} \hat{q}_0(\alpha k) + (\alpha^2 - 1)k^2 G_0(ik^3) + i(1-\alpha)kG_1(ik^3), & k \in D_1^+, \\ \hat{q}_0(\alpha^2 k) + (\alpha - 1)k^2 G_0(ik^3) + i(1-\alpha^2)kG_1(ik^3), & k \in D_2^+, \end{cases} \tag{1.51}$$

with $\alpha = \exp[2i\pi/3]$. Furthermore $G_0(ik^3)$ and $G_1(ik^3)$ can be replaced by $G_0(ik^3, t)$ and $G_1(ik^3, t)$ defined in (1.9).

Indeed, replacing $\tilde{g}_0$ and $\tilde{g}_1$ by G_0 and G_1 in (1.49) and adding the resulting expressions to the expression for $\tilde{g}(k)$ we find (1.51).

Example 1.12 (the first Stokes equation). For (1.24a), $\alpha_n = -i$, and thus $N = 1$. Using

$$w(k) - w(\nu) = (ik - ik^3) - (i\nu - i\nu^3) = i(k-\nu)(1 - k^2 - \nu^2 - k\nu),$$

it follows that

$$\nu_j^2 + k\nu_j + k^2 - 1 = 0, \quad \nu_j \sim e^{\frac{2ij\pi}{3}}k, \quad k \to \infty, \quad j = 1, 2. \tag{1.52}$$

The domains D_R^+, $D_{R_1}^-$, $D_{R_2}^-$ coincide with the domain, D^+, D_1^-, D_2^- of Figure 1.3. If $k \in D_R^+$, then $\nu_1 \in D_{R_1}^-$ and $\nu_2 \in D_{R_2}^-$. Thus we consider the global relation (1.18) with $\tilde{g}$ given by (1.24d), i.e.,

$$(k^2 - 1)\tilde{g}_0 - ik\tilde{g}_1 - \tilde{g}_2 = \tilde{g}(k),$$

and we replace k with ν_1 and ν_2:

$$\hat{q}_0(\nu_1) + (1 - \nu_1^2)\tilde{g}_0 + i\nu_1\tilde{g}_1 + \tilde{g}_2 = e^{w(k)T}\hat{q}_T(\nu_1), \tag{1.53a}$$

$$\hat{q}_0(\nu_2) + (1 - \nu_2^2)\tilde{g}_0 + i\nu_2\tilde{g}_1 + \tilde{g}_2 = e^{w(k)T}\hat{q}_T(\nu_2), \quad k \in D^+. \tag{1.53b}$$

The Dirichlet Problem. The solution $q(x,t)$ is given by (1.16), where $w(k) = ik - ik^3$, the contour in the second term of (1.16) is either ∂D^+ of Figure 1.5 or the contour ∂D_R^+ corresponding to ∂D^+, which is depicted in Figure 1.3, and $\tilde{g}(k)$ is given by

$$\tilde{g}(k) = \frac{1}{\nu_1 - \nu_2}\left[(\nu_1 - k)\hat{q}_0(\nu_2) + (k - \nu_2)\hat{q}_0(\nu_1)\right] + (3k^2 - 1)G_0(w(k)), \quad k \in D^+. \tag{1.54}$$

Furthermore $G_0(w(k))$ can be replaced by $G_0(w(k), t)$.

Indeed we replace $\tilde{g}_0$ by $\tilde{G}_0$ in the homogeneous versions of (1.53) and also supplement these equations with the equation for $\tilde{g}(k)$ where $\tilde{g}_0$ is replaced with $\tilde{G}_0$. In order to eliminate the unknown functions $\tilde{g}_1$ and $\tilde{g}_2$ from these three equations, we multiply the homogeneous versions of (1.53a) and (1.53b) by α_1 and α_2, respectively, and then add the resulting expressions to the equation for $\tilde{g}(k)$; the coefficients of $\tilde{g}_1$ and $\tilde{g}_2$ vanish, provided that

$$\alpha_1 + \alpha_2 = 1, \quad \alpha_1\nu_1 + \alpha_2\nu_2 = k, \quad \text{i.e.,} \quad \alpha_1 = \frac{k - \nu_2}{\nu_1 - \nu_2}, \quad \alpha_2 = \frac{\nu_1 - k}{\nu_1 - \nu_2}$$

and then (1.54) follows by making use of the identities

$$\nu_1 + \nu_2 = k, \quad \nu_1\nu_2 = k^2 - 1.$$

Example 1.13 (the second Stokes equation). For (1.25a), $\alpha_n = i$, and thus $N = 2$. The functions ν_1 and ν_2 satisfy

$$\nu_j^2 + k\nu_j + k^2 + 1 = 0, \quad j = 1, 2, \quad \nu_j \sim e^{\frac{2ij\pi}{3}}, \quad k \to \infty, \quad j = 1, 2. \tag{1.55}$$

The domains $D_{R_1}^+$, $D_{R_2}^+$, D_R^- are as depicted in Figure 1.4. If $k \in D_{1R}^+$, then $\nu_1 \in D_R^-$ and if $k \in D_{2R}^+$, then $\nu_2 \in D_R^-$. Thus in order to determine $\tilde{g}(k)$ for $k \in D_{1R}^+$ and $k \in D_{2R}^+$ we must use ν_1 and ν_2, respectively. Hence we consider the global relation (1.18) with $\tilde{g}$ given by (1.25d), i.e.,

$$-(1 + k^2)\tilde{g}_0 + ik\tilde{g}_1 + \tilde{g}_2 = \tilde{g}(k),$$

and replace k with either ν_1 or ν_2:

$$\hat{q}_0(\nu_1) + (1 + \nu_1^2)\tilde{g}_0 - i\nu_1\tilde{g}_1 - \tilde{g}_2 = e^{w(k)T}\hat{q}_T(\nu_1), \quad k \in D_{1R}^+, \tag{1.56a}$$

$$\hat{q}_0(\nu_2) + (1 + \nu_2^2)\tilde{g}_0 - i\nu_2\tilde{g}_1 - \tilde{g}_2 = e^{w(k)T}\hat{q}_T(\nu_2), \quad k \in D_{2R}^+. \tag{1.56b}$$

The Canonical Problem. In this case $q(0, t)$ and $q_x(0, t)$ are given; see (1.50). The solution $q(x, t)$ is given by (1.29) where $w(k) = ik + ik^3$, the curves ∂D_{R1}^+ and ∂D_{R2}^+ are the curves ∂D_1^+ and ∂D_2^+ of Figure 1.4, and $\tilde{g}(k)$ is given by

$$\tilde{g}(k) = \begin{cases} \hat{q}_0(\nu_1) + (\nu_1^2 - k^2)G_0(w(k)) + i(k - \nu_1)G_1(w(k)), & k \in D_{1R}^+, \\ \hat{q}_0(\nu_2) + (\nu_2^2 - k^2)G_0(w(k)) + i(k - \nu_2)G_1(w(k)), & k \in D_{2R}^+. \end{cases} \tag{1.57}$$

Furthermore, $G_0(w(k))$ and $G_1(w(k))$ can be replaced by $G_0(w(k), t)$ and $G_1(w(k), t)$.

Indeed, replacing the functions $\tilde{g}_0$ and $\tilde{g}_1$ by G_0 and G_1 in the expression for $\tilde{g}(k)$ and adding the resulting expression to the homogeneous versions of (1.56), we find (1.57).

Example 1.14 (a PDE with a fifth order derivative). For (1.26a), $\alpha_n = -i$, and thus $N = 2$. The functions $\{\nu_j\}_1^4$ satisfy

$$\nu_j^4 + k\nu_j^3 + k^2\nu_j^2 + k^3\nu_j + k^4 - 1 = 0, \quad \nu_j \sim e^{\frac{2ji}{5}}k, \quad k \to \infty, \quad j = 1, 2, 3, 4. \tag{1.58}$$

The domains $D_{R_1}^+$, $D_{R_2}^+$, $D_{R_1}^-$, $D_{R_2}^-$, $D_{R_3}^-$ are as depicted in Figure 1.7(b). These domains and the definitions (1.58) imply the following:

$$k \in D_{R_1}^+ : \nu_1 \in D_{R_1}^-, \quad \nu_2 \in D_{R_2}^-, \quad \nu_3 \in D_{R_3}^-$$

and

$$k \in D_{R_2}^+ : \nu_2 \in D_{R_1}^-, \quad \nu_3 \in D_{R_2}^-, \quad \nu_4 \in D_{R_3}^-.$$

The Canonical Problem. In this case $q(0, t)$ and $q_x(0, t)$ are given; see (1.50).

We consider the global relation (1.18) with $\tilde{g}$ given by (1.26d), i.e.,

$$(k^4 - 1)G_0 - ik^3G_1 - k^2\tilde{g}_2 + ik\tilde{g}_3 + \tilde{g}_4 = \tilde{g}(k). \tag{1.59}$$

For $k \in D_{R_1}^+$ we can determine $\tilde{g}(k)$ by eliminating $\{\tilde{g}_j\}_2^4$ from the above equation, and from the equations obtained from the homogeneous version of the global relation, by replacing k with $\{\nu_j\}_1^3$, i.e., from the equations

$$\hat{q}_0(\nu_j) + (1 - \nu_j^4)G_0 + i\nu_j^3G_1 + \nu_j^2\tilde{g}_2 - i\nu_j\tilde{g}_3 - \tilde{g}_4, k \in D_{R_1}^+, \quad j = 1, 2, 3. \tag{1.60}$$

For $k \in D^+_{R_2}$ we can determine $\tilde{g}(k)$ by eliminating $\{\tilde{g}_j\}_2^4$ from (1.59) and from an equation similar to (1.60) but with $j = 2, 3, 4$. Then we insert these expressions in (1.29) where $w(k) = ik - ik^5$ and $\partial D^+_{R_1}$, $\partial D^+_{R_2}$ are as depicted in Figure 1.7(b).

1.1 The Classical Representations: Return to the Real Line

It was mentioned in the introduction that the global relation, and the equations obtained from the global relation through the transformations which leave $w(k)$ invariant, provide an alternative approach to deriving the classical representations. As an example, consider the heat equation (1.20a). In this case the global relation and the equation obtained through the transformation $k \to -k$ are (1.35a) and (1.35b). Replacing T by t in these equations we find

$$\hat{q}_0(k) - \int_0^t e^{k^2 s}[ikq(0,s) + q_x(0,s)]ds = e^{k^2 t}\hat{q}(k,t), \quad \operatorname{Im} k \le 0, \tag{1.61a}$$

$$\hat{q}_0(-k) - \int_0^t e^{k^2 s}[-ikq(0,s) + q_x(0,s)]ds = e^{k^2 t}\hat{q}(-k,t), \quad \operatorname{Im} k \ge 0, \tag{1.61b}$$

where $\hat{q}_0(k)$ and $\hat{q}(k,t)$ denote the Fourier transform of $q_0(x)$ and $q(x,t)$; see (1.6) and (3). Equations (1.61) are both valid for k real. Thus, the algebraic manipulation of these equations yields an integral transform for $q(x,t)$ with k on the real line.

(i) *The Dirichlet Problem.* In this case the function $q(0,t)$ is known, and thus we must eliminate $q_x(0,t)$. Subtracting equations (1.61) we find the sine transform of $q(x,t)$; see (18).

(ii) *The Neumann Problem.* In this case $q_x(0,t)$ is known, and thus by adding equations (1.61) we find

$$\int_0^\infty q(x,t)\cos(kx)dx = e^{-k^2 t}\left[\int_0^\infty q_0(x)\cos(kx)dx - G_1(k^2,t)\right], \quad k \in \mathbb{R},$$

where $G_1(k,t)$ is the t-transform of the Neumann data involving an integral from 0 to t; see (1.9).

(iii) *The Robin Problem.* Let q satisfy the Robin boundary condition (1.38a). In this case after using this condition to express $q_x(0,s)$ in terms of $q(0,s)$, equations (1.61) become two equations involving the unknown $q(0,s)$. Eliminating this unknown function we find

$$\int_0^\infty \left(e^{-ikx} + \frac{k - i\gamma}{k + i\gamma}e^{ikx}\right) q(x,t)dx = e^{-k^2 t}\left[\hat{q}_R(k) - \frac{2k}{k + i\gamma}G_R(k^2,t)\right], \quad k \in \mathbb{R}, \tag{1.62a}$$

where

$$\hat{q}_R(k) = \int_0^\infty \left(e^{-ikx} + \frac{k - i\gamma}{k + i\gamma}e^{ikx}\right) q_0(x)dx, \quad k \in \mathbb{R}, \tag{1.62b}$$

$$G_R(k,t) = \int_0^t e^{ks} g_R(s)ds, \quad k \in \mathbb{C}. \tag{1.62c}$$

The above approach has the advantage that it avoids the determination of the "proper" transform (it also avoids integration by parts). For example, the fact that the proper transform for the Dirichlet problem is the sine transform is a direct consequence of (1.61). However, the above approach has the disadvantage that it requires the independent knowledge of how to invert the integral transforms of $q(x,t)$. This of course can be achieved using results from the Sturm–Liouville theory. Alternatively, one can obtain the classical integral representations without using the Sturm-Liouville theory by starting with the novel integral representations and then using contour deformation.

For example, for the Dirichlet problem the novel integral representation is given either by (1.36) or by (16). Regarding the latter equation we note that $\hat{q}_0(-k)$ is analytic in $\mathbb{C}^+$, whereas $\exp[-k^2t]kG_0(k^2,t)$ is bounded and analytic in $\mathbb{C}^+\backslash D^+$. Thus the contour ∂D^+ can be deformed back to the real line,

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^2t}\hat{q}_0(k)dk - \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^2t}[\hat{q}_0(-k) + 2ikG_0(k^2,t)]dk.$$

By splitting the integral along the real line into an integral from $-\infty$ to 0 and an integral from 0 to ∞, the above expression becomes (19).

Similarly, for the Robin problem the novel integral representation is given by (1.38b). Replacing $G_R(k^2)$ and $G_R(-\gamma^2)$ by $G_R(k^2,t)$ and $G_R(-\gamma^2,t)$ in this equation, deforming the contour ∂D^+ back to the real line, and splitting the integral along the real line into the two integrals mentioned earlier, we find that if $\gamma > 0$, then

$$q(x,t) = \frac{1}{2\pi}e^{-k^2t}\left\{\int_0^{\infty}\left[e^{ikx}\hat{q}_R(k) + e^{-ikx}\hat{q}_R(-k)\right]dk\right.$$

$$\left.-2\int_0^{\infty}\left(\frac{e^{-ikx}}{k+i\gamma} + \frac{e^{ikx}}{k-i\gamma}\right)kG_R(k^2,t)dk\right\}, \quad 0 < x < \infty, \quad t > 0. \tag{1.63}$$

Remark 1.4. It is *always* possible to deform from the real line to the contour ∂D^+ *before* using the global relation to eliminate the unknown boundary values. However, it is in general *not* possible to deform back to the real line *after* using the global relation. Actually, this "return to the real line" is possible only in the exceptional cases that there exists a classical x-transform pair. Let us consider, for example, the novel integral representations for the Dirichlet problem of the heat equation and of the PDE (1.21a). These representations involve $\hat{q}_0(-k)$ and $\hat{q}_0(-k+i\beta)$, respectively, and thus while $\hat{q}_0(-k)$ is defined in the entire upper half of the complex k-plane, the domain of definition of $\hat{q}_0(-k+i\beta)$ is restricted:

$$\hat{q}_0(-k+i\beta) = \int_0^{\infty} e^{i(k-i\beta)x}q_0(x)dx, \quad \operatorname{Im} k \geq \beta. \tag{1.64}$$

Hence, since this term is *not* bounded for $\operatorname{Im} k < \beta$ (unless $q_0(x)$ decays exponentially), we *cannot* deform the contour ∂D^+ of Figure 1.2 back to the real line. This indicates that there does *not* exist for this problem an x-transform formulated on the real k-axis. This is consistent with the following argument: Using the transformation $u = q\exp[\beta x/2]$, (1.21a) is mapped to the equation

$$u_t = u_{xx} - \frac{\beta^2}{4}u,$$

which *can* be solved by classical transforms. However, the initial condition for u is given by $u_0(x) = q_0(x)\exp[\beta x/2]$, and thus this approach works only for exponentially decaying initial conditions.

The situation for the PDEs involving a third and a fifth order derivative, discussed in Examples 1.10–1.14, is similar to that of (1.21a).

1.2 Forced Problems

Let $q(x,t)$ satisfy the inhomogeneous PDE

$$q_t + w(-i\partial_x)q = f(x,t) \tag{1.65}$$

in the domain Ω where $f(x,t)$ has appropriate smoothness and decay.

Equation (1.65) can be written in a form similar with (43), where now the RHS of (43) is replaced with $\exp[-ikx + w(k)t]f$. This term yields the following additional term in the RHS of (44):

$$F(k,t) = \int_0^t \left(\int_0^\infty e^{-ik\xi + w(k)\tau} f(\xi,\tau)d\xi \right) d\tau. \tag{1.66}$$

This implies that $q(x,t)$ satisfies an equation similar with (1.29) where $\hat{q}_0(k)$ is now replaced by

$$\hat{q}_0(k) + F(k,t).$$

Remark 1.5. Nonlinear evolution PDEs can be considered as forced linear PDEs. By using the explicit formulae for forced linear PDEs derived by the new method, it should be possible to study the well-posedness of a large class of nonlinear PDEs. This should yield existence at least for small time, or for boundary conditions which have small norms in an appropriate function space.

1.3 Green's Function Type Representations

The integral representation of the solution of the PDE (1.1) formulated on the half-line is given by (1.29) where the function $\tilde{g}(k)$ involves integrals of the given initial and boundary conditions. Furthermore, the integrals from 0 to T of the boundary conditions can be replaced by integrals from 0 to t. Thus, by changing the order of the k-integration with the x- and t-integrations, it is *always* possible to express the solution in the form (20) of the introduction.

Second order PDEs are distinguished by the fact that for simple boundary value problems, $G^{(I)}$ and $G^{(B)}$ can be computed *explicitly*.

As an illustration, we will derive $G^{(I)}$ and $G^{(B)}$ for the Neumann problems of the heat equation and for the Dirichlet problem of the first Stokes equation.

1.3.1. The Heat Equation with Neumann Boundary Conditions

The solution is given by (1.37), where we replace $G_1(k^2)$ by $G_1(k^2,t)$. Hence,

$$q(x,t) = \int_0^\infty G^{(I)}(x,t,\xi)q_0(\xi)d\xi + \int_0^t G^{(B)}(x,t,s)g_1(s)ds, \tag{1.67}$$

where

$$G^{(I)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-\xi)-k^2 t}dk + \frac{1}{2\pi}\int_{\partial D^+} e^{ik(x+\xi)-k^2 t}dk, \tag{1.68a}$$

$$G^{(B)} = -\frac{1}{\pi}\int_{\partial D^+} e^{ikx-k^2(t-s)}dk. \tag{1.68b}$$

The integrals defined by (1.68) can be computed explicitly:

$$G^{(I)}(x,t,\xi) = \frac{1}{2\sqrt{\pi t}}\left[e^{-\frac{(x-\xi)^2}{4t}} + e^{-\frac{(x+\xi)^2}{4t}}\right], \quad 0 < x < \infty, \quad 0 < \xi < \infty, \quad t > 0, \tag{1.69a}$$

$$G^{(B)}(x,t,s) = -\frac{e^{-\frac{x^2}{4(t-s)}}}{\sqrt{\pi(t-s)}}, \quad 0 < x < \infty, \quad t > 0, \quad 0 < s < t. \tag{1.69b}$$

Indeed, we first consider the first integral in the RHS of (1.68a) which we call I. By completing the square of the exponential of this integral we find

$$I = e^{-\frac{(x-\xi)^2}{4t}}\int_{-\infty}^{\infty} e^{-t(k-i\eta)^2}dk, \quad \eta = \frac{x-\xi}{2t}.$$

But

$$\int_{-\infty}^{\infty} e^{-t(k-i\eta)^2}dk = \int_{-\infty-i\eta}^{\infty-i\eta} e^{-tz^2}dz = \int_{-\infty}^{\infty} e^{-tz^2}dz = \frac{1}{\sqrt{t}}\int_{-\infty}^{\infty} e^{-l^2}dl = \frac{\sqrt{\pi}}{\sqrt{t}},$$

where the first equality follows from the change of variables $z = k - i\eta$, the second equality follows from Cauchy's theorem, and the third from the change of variables $l = \sqrt{t}z$. Hence, the evaluation of I yields the first term in the expression for $G^{(I)}$. Next we consider the second integral in the RHS of (1.68a) and the integral in (1.68b): The contour ∂D^+ can be replaced by the real line and then these two integrals can be mapped to the integral I by the transformations $\xi \to -\xi$ and $\{t \to t - s, \xi \to 0\}$. Hence, these integrals yield the second terms in $G^{(I)}$ and $G^{(B)}$.

1.3.2. The First Stokes Equation with Dirichlet Boundary Conditions

The solution is given by (1.16), where $w(k) = ik - ik^3$, ∂D^+ is as depicted in Figure 1.5, and $\tilde{g}(k)$ is as defined in (1.54). Hence, replacing in the latter equation $\tilde{G}_0(w(k))$ by $\tilde{G}_0(w(k), t)$ we find that

$$G^{(I)}(x,t,\xi), \quad 0 < x < \infty, \quad 0 < \xi < \infty, \quad t > 0,$$

and

$$G^{(B)}(x,t,s), \quad 0 < x < \infty, \quad t > 0, \quad 0 < s < t,$$

are given by

$$G^{(I)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-\xi)-i(k-k^3)t}dk$$

$$-\frac{1}{2\pi(\nu_1-\nu_2)}\int_{\partial D^+}\left[(\nu_1-k)e^{i(kx-\nu_2\xi)-i(k-k^3)t}+(k-\nu_2)e^{i(kx-\nu_1\xi)-i(k-k^3)t}\right]dk, \quad (1.70a)$$

$$G^{(B)} = -\frac{1}{2\pi(\nu_1-\nu_2)}\int_{\partial D^+}(3k^2-1)e^{-i(k^3-k)(t-s)+ikx}dk. \qquad (1.70b)$$

The above integrals cannot be computed explicitly. However, in the case that the term q_x is missing in the second Stokes equation, i.e., in the case of (1.22a), $G^{(B)}$ can be expressed in terms of the Airy function.

1.4 The Generalized Dirichlet to Neumann Correspondence

The global relation (1.18) involves the t-integrals of the boundary values $\{\partial_x^j q(0,t)\}_0^{n-1}$, the coefficients $\{c_j(k)\}_0^{n-1}$ (which are uniquely determined in terms of $w(k)$ by (1.15)), the Fourier transform $\hat{q}_0(k)$ of the initial condition $q_0(x)$, and the Fourier transform $\hat{q}(k,T)$ of $q(x,T)$. It turns out that using the global relation and the equations obtained from the global relation through the transformations that leave $w(k)$ invariant, it is possible to determine the unknown boundary values directly *without* having to determine $q(x,t)$. In the following we present some illustrative examples.

1.4.1. The Neumann to Dirichlet Map for the Heat Equation

For brevity of presentation we let $q_0(x)=0$. Equation (1.35a) with $\hat{q}_0(k)=0$ becomes

$$-\int_0^T e^{k^2 s}q_x(0,s)ds + ik\int_0^T e^{k^2 s}q(0,s)ds = e^{k^2 T}\int_0^\infty e^{ikx}q(x,T)dx, \quad \operatorname{Im} k \ge 0. \qquad (1.71)$$

We multiply this equation by $\exp[-k^2 t]$ and integrate the resulting equation along the contour ∂D^+ depicted in Figure 1.1. The term $\exp[k^2(T-t)]$ is bounded and analytic in the domain D^+ and the x-Fourier transform of $q(x,T)$ is analytic in $\mathbb{C}^+$ and of $0(1/k)$ as $k\to\infty$, and thus the RHS of (1.71) yields a zero contribution. Furthermore, the change of variables $k^2=il$ and the classical Fourier transform formula imply that the second term on the LHS of (1.71) yields $-\pi q(0,t)$. Hence, (1.71) implies

$$\int_{\partial D^+}\left[\int_0^T e^{k^2(s-t)}q_x(0,s)ds\right]dk + \pi q(0,t) = 0. \qquad (1.72)$$

We split the integral $\int_0^T$ into $\int_0^t$ and $\int_t^T$. The second integral vanishes due to the fact that the integrand is analytic in D^+, whereas the first integral can be simplified as follows:

$$\int_{\partial D^+}\left[\int_0^t e^{-k^2(t-s)}q_x(0,s)ds\right]dk = \int_{\partial D^+}\left[\int_0^t e^{-l^2}\frac{q_x(0,s)ds}{\sqrt{t-s}}\right]dl$$
$$= c\int_0^t \frac{q_x(0,s)ds}{\sqrt{t-s}}, \qquad (1.73)$$

with

$$c = \int_{\partial D^+} e^{-l^2} dl = \sqrt{\pi},$$

where the first equality in (1.73) follows from the change of variables $l = k\sqrt{t-s}$ and the evaluation of c follows from the fact that $\exp[-z^2]$ is bounded in $E^+ = \mathbb{C}^+/D^+$, and thus the contour ∂D^+ can be deformed to the real line. Hence, (1.72) yields (23).

1.4.2. The Dirichlet to Neumann Map for the Heat Equation

Using the Abel transform, it is possible to solve (23) for $q_x(0,t)$ in terms of $q(0,t)$. Alternatively, we multiply (1.71) by $ik\exp[-k^2t]$ and integrate the resulting equation along ∂D^+. The RHS of (1.71) yields a zero contribution, whereas the first term on the LHS of (1.71) yields $\pi q(0,t)$. Before changing the order of integration in the second integral on the LHS of (1.71) we must first integrate by parts:

$$-k^2 \int_0^T e^{k^2(s-t)} q(0,s)ds = q(0,0)e^{-k^2t} - e^{k^2(T-t)} q(0,T) + \int_0^T e^{k^2(s-t)} \dot{q}(0,s)ds, \quad (1.74)$$

where the dot denotes differentiation. We recall that $q(x,0) = 0$, thus $q(0,0) = 0$, and furthermore the integral of $\exp[k^2(T-t)]$ along ∂D^+ vanishes. The integral along ∂D^+ of the third term in the RHS of (1.74) was computed in (1.73), and hence (1.71) yields

$$q_x(0,t) = -\frac{1}{\sqrt{\pi}} \int_0^t \frac{\dot{q}(0,s)ds}{\sqrt{t-s}}, \quad t > 0. \quad (1.75)$$

1.4.3. A PDE with a Third Order Derivative

Let $q(x,t)$ satisfy (1.22a). We assume that $q_0(x) = 0$ and we will express $q(0,t)$ and $q_x(0,t)$ in terms of $q_{xx}(0,t)$. Solving the global relations (1.46) (where $\hat{q}_0 = 0$) for $\tilde{g}_0$ and $\tilde{g}_1$ we find

$$k^2\tilde{g}_0 + \tilde{g}_2 = \frac{e^{-ik^3T}}{\alpha - 1}\left[\alpha\hat{q}(\nu_1, T) - \hat{q}(\nu_2, T)\right], \quad (1.76a)$$

$$ik^2\tilde{g}_1 - k\tilde{g}_2 = \frac{ke^{-ik^3T}}{\alpha - 1}\left[\hat{q}(\nu_1, T) - \alpha\hat{q}(\nu_2, T)\right]. \quad (1.76b)$$

We multiply (1.76a) and (1.76b) by $\exp[ik^3t]$ and integrate the resulting equations along the curve ∂D^+ which is depicted in red in Figure 1.3. The RHSs of equations (1.76) yield a zero contribution, whereas the first terms on the LHSs of (1.76a) and (1.76b) yield $-2\pi q(0,t)/3$ and $-2i\pi q_x(0,t)/3$, respectively. Hence, equations (1.76) imply

$$-\frac{2\pi}{3}q(0,t) + \int_{\partial D^+}\left[\int_0^T e^{ik^3(t-s)} q_{xx}(0,s)ds\right]dk = 0, \quad 0 < t < T,$$

$$-\frac{2i\pi}{3}q_x(0,t) - \int_{\partial D^+} k\left[\int_0^T e^{ik^3(t-s)} q_{xx}(0,s)ds\right]dk = 0, \quad 0 < t < T.$$

The integrals $\int_t^T$ yield a zero contribution, whereas the integrals $\int_0^t$ can be simplified as follows:

$$\int_{\partial D^+}\left[\int_0^t e^{ik^3(t-s)}q_{xx}(0,s)ds\right]dk = \int_{\partial D^+}\left[\int_0^t \frac{e^{il^3}q_{xx}(0,s)}{(t-s)^{\frac{1}{3}}}ds\right]dl$$
$$= c_1\int_0^t \frac{q_{xx}(0,s)ds}{(t-s)^{\frac{1}{3}}},$$
$$\int_{\partial D^+}k\left[\int_0^t e^{ik^3(t-s)}q_{xx}(0,s)ds\right]dk = \int_{\partial D^+}\left[\int_0^t \frac{e^{il^3}q_{xx}(0,s)}{(t-s)^{\frac{2}{3}}}ds\right]ldl$$
$$= c_2\int_0^t \frac{q_{xx}(0,s)ds}{(t-s)^{\frac{2}{3}}},$$

where $l = k(t-s)^{\frac{1}{3}}$ and the constants c_1 and c_2 are defined by

$$c_1 = \int_{\partial D^+} e^{il^3}dl, \quad c_2 = \int_{\partial D^+} le^{il^3}dl. \tag{1.77}$$

Thus

$$q(0,t) = \frac{3c_1}{2\pi}\int_0^t \frac{q_{xx}(0,s)}{(t-s)^{\frac{1}{3}}}ds, \quad t>0,$$
$$q_x(0,t) = \frac{3ic_2}{2\pi}\int_0^t \frac{q_{xx}(0,s)}{(t-s)^{\frac{2}{3}}}ds, \quad t>0. \tag{1.78}$$

The constants c_1 and c_2 can be expressed in terms of the Gamma function.

1.5 Rigorous Considerations

Proposition 1.2 was derived under the assumption of existence. However, it is possible to prove the relevant result *without* the a priori assumption of existence. This proof is straightforward provided that the initial and boundary conditions have sufficient smoothness and decay. Consider for example the Dirichlet problem for the heat equation. Given $q_0(x)$ and $g_0(t)$ we define $\hat{q}_0(k)$ and $G_0(k)$ by (1.6) and (1.8) (with $j = 0$) and then define $q(x,t)$ by (1.36). We must now prove that this function satisfies the heat equation and that $q(x,0) = q_0(x)$, $q(0,t) = g_0(t)$. The relevant integrands depend on x and t only through $\exp[ikx - k^2t]$, which immediately implies that $q(x,t)$ satisfies the heat equation. In order to prove that $q(x,0) = g_0(x)$ we evaluate (1.36) at $t = 0$:

$$q(x,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\hat{q}_0(k)dk - \frac{1}{2\pi}\int_{\partial D^+} e^{ikx}\left[\hat{q}_0(-k) + 2ikG_0(k^2)\right]dk.$$

The second integral on the RHS of this equation vanishes due to analyticity considerations, and the first integral equals $q_0(x)$ due to the usual Fourier transform identity. In order to prove that $q(0,t) = g_0(t)$ we evaluate (1.36) at $x = 0$:

$$q(0,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-k^2 t}\hat{q}_0(k)dk - \frac{1}{2\pi}\int_{\partial D^+} e^{-k^2 t}\hat{q}_0(-k)dk \\ -\frac{i}{\pi}\int_{\partial D^+} e^{-k^2 t}G_0(k^2)kdk. \tag{1.79}$$

The contour ∂D^+ of the second integral on the RHS of this equation can be deformed to the real k-axis, and then the transformation $k \to -k$ yields an integral which cancels with the first integral of (1.79). Using the transformation $k^2 \to il$ in the third integral in the RHS of (1.79), and employing the usual Fourier transform identity, we find that this integral equals $g_0(t)$.

Uniqueness can be established using the usual PDE techniques. These arguments are sufficient to establish well-posedness provided that the functions $q_0(x)$ and $g_0(t)$ have "sufficient smoothness" and the function $q_0(x)$ has "sufficient decay." A precise characterization of a suitable function space is given in [77] for the case that $\operatorname{Re} w(k) = 0$ for k real.

Theorem 1.1 (existence and uniqueness of a weak solution for Sobolev data [77]). Assume that

- q_0 belongs to the Sobolev space $H^{\tilde{n}}(0,\infty)$, where $\tilde{n}$ is the smallest integer $\geq n/2$;
- g_l belongs to the Sobolev space $H^{\frac{1}{2}+\frac{(2\tilde{n}-2l-1)}{2n}}(0,T)$ for $0 \leq l \leq N-1$;
- $g_l(0) = \partial_x^l q_0(0)$ for $0 \leq l \leq N-1$.

Then there exists a unique function $q(x,t)$ with the following properties:

- $t \to q(\cdot,t)$ is a continuous map from $[0,T]$ into $H^{\tilde{n}}(0,\infty)$;
- $q(x,t)$ satisfies the initial and boundary conditions (1.27) and (1.28).

Given any $\phi \in C_c^{\infty}(\mathbb{R})$ such that $\partial_x^j\phi(0) = 0$ for all $0 \leq j \leq n-\tilde{n}-1$, the function $(q(\cdot,t),\phi)_{L_2(0,\infty)}$ is differentiable on $(0,T)$.

The map $x \to \partial_x^j q(x,\cdot)$ is a continuous map from $[0,\infty)$ into $H^{\frac{1}{2}+\frac{2\tilde{n}-2j-1}{2n}}(0,T)$ for $0 \leq j \leq n-1$.

The proof of the above result can be found in [77]. In what follows we use a simple example to explain the reason for the appearance of $H^{\tilde{n}}$. Let $q(x,t)$ solve the Dirichlet problem for the Schrödinger equation with zero potential, i.e., (72) of the introduction.

In this case

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-ik^2 t}\hat{q}_0(k)dk - \frac{1}{2\pi}\int_{\partial D^+} e^{ikx-ik^2 t}[\hat{q}_0(-k) + kG_0(-ik^2)]dk,$$

where $\hat{q}_0(k)$ is the Fourier transform of the initial condition $q_0(x)$, ∂D^+ is the oriented boundary of the first quadrant of the complex k-plane, and $G_0(k)$ is the t-transform of the Dirichlet boundary condition $g_0(t)$. Let $G_0(-ik^2) = G_T(k)$, i.e.,

$$G_T(k) = \int_0^T e^{ik^2 s} g_0(s)ds, \quad k \in \mathbb{C}.$$

The terms involving $\hat{q}_0(\pm k)$ can be analyzed by the usual theory of Fourier transforms, so we let $q_0(x) = 0$. In this case G_0 must belong to $H_0^{\frac{3}{4}}(0, T)$, i.e.,

$$g_0 \in H^{\frac{3}{4}}(0, T) \text{ and } g_0(0) = g_0(T) = 0.$$

Indeed, using

$$\left(1 + |\tau|^{\frac{3}{4}}\right) G_T(\sqrt{\tau}) \in L_2(-\infty, \infty),$$

it follows that

$$(1 + |k|)(k G_T(k)) \in L_2(\partial D^+).$$

Hence, the map

$$t \to f_1(x, t) = \int_0^{\infty} e^{ikx - ik^2 t} k G_T(k) dk$$

defines a continuous map from $[0, T]$ to $L_2([0, \infty))$. Similarly, the map

$$t \to f_2(x, t) = \int_{i\infty}^{0} e^{ikx - ik^2 t} k G_T(k) dk$$

also defines a continuous map from $[0, T]$ to $L_2([0, \infty))$. This is a consequence of a classical result in [78] which states that the map

$$u(x) \to \int_0^{\infty} e^{ikx} u(x) dx$$

defines a bounded linear map from $L_2(0, \infty)$ to $L_2(i\infty, 0)$.

Chapter 2

Evolution Equations on the Finite Interval

In this chapter we will consider (1.1) in the finite interval $0 < x < L$, where L is a positive finite constant.

We first introduce some useful notations:

- An initial-boundary value problem for (1.1) on the positive finite interval is well posed if N boundary conditions are presented at $x = 0$ and $n - N$ boundary conditions are presented at $x = L$, where the integer N is as defined in (1.5).
- The function $\hat{q}_0(x)$ will still denote the Fourier transform of the initial condition,

$$q(x, 0) = q_0(x), \;\; 0 < x < L; \;\; \hat{q}_0(k) = \int_0^L e^{-ikx} q_0(x) dx, \quad k \in \mathbb{C}. \tag{2.1a}$$

  It is assumed that $q_0(x)$ has sufficient smoothness. Similarly $\hat{q}_{\scriptscriptstyle T}(k)$ will denote the Fourier transform of $q(x, T)$,

$$\hat{q}_{\scriptscriptstyle T}(k) = \int_0^L e^{-ikx} q(x, T) dx, \quad k \in \mathbb{C}. \tag{2.1b}$$

- In analogy with (1.7), the t-transform of the boundary values at $x = L$ will be denoted by $\tilde{h}_j$,

$$\tilde{h}_j(k) = \int_0^T e^{ks} \partial_x^j q(L, s) ds, \;\; k \in \mathbb{C}, \;\; j = 0, 1, \ldots, n - 1. \tag{2.2}$$

- In analogy with (1.8), if the boundary value $\partial_x^j q(L, t)$ is prescribed as a boundary condition, then it will be denoted by $h_j(t)$ and its t-transform by $H_j(k)$,

$$\partial_x^j q(L, t) = h_j(t), \;\; 0 < t < T; \;\; H_j(k) = \int_0^T e^{ks} h_j(s) ds, \quad k \in \mathbb{C}. \tag{2.3}$$

  It is assumed that $h_j(t)$ has sufficient smoothness and that it is compatible with $q_0(x)$ at $x = L, t = 0$, i.e., $h_j(0) = (\frac{d}{dx})^j q_0(L)$.

- If q and its first $N-1$ derivatives are prescribed at $x=0$ and q and its first $N-n-1$ derivatives are prescribed at $x=L$, then we will refer to this problem as the *canonical* problem.
- It will turn out that the integral representation which will be derived for q is also valid if T is replaced by t in (1.8) and (2.2). In analogy with (1.9) the relevant integral will be denoted by $H_j(k,t)$, i.e.,

$$H_j(k,t) = \int_0^t e^{ks} h_j(s)ds, \quad k \in \mathbb{C}, \quad 0 < t < T. \tag{2.4}$$

We next present the analogue of Proposition 1.1.

Proposition 2.1 (a general integral representation). Let $q(x,t)$ satisfy the linear evolution PDE (1.1) in the domain

$$\Omega_L = \{0 < x < L, 0 < t < T\}, \tag{2.5}$$

where $w(k)$ satisfies the restrictions specified in (1.2) and L, T are finite positive constants.

Assume that $q(x,t)$ is a sufficiently smooth (up to the boundary Ω_L) solution of (1.1). Then $q(x,t)$ is given by

$$\begin{aligned} q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-w(k)t}\hat{q}_0(k)dk - \frac{1}{2\pi}\int_{\partial D^+} e^{ikx-w(k)t}\tilde{g}(k)dk \\ -\frac{1}{2\pi}\int_{\partial D^-} e^{-ik(L-x)-w(k)t}\tilde{h}(k)dk, \quad (x,t)\in\Omega_L, \end{aligned} \tag{2.6}$$

where ∂D^+, ∂D^- are the oriented boundaries of the domains D^+ and D^- defined in (1.11), $\tilde{g}(k)$ is defined in terms of the t-transforms of the boundary values $\{\partial_x^j q(0,t)\}_0^{n-1}$ by (1.17), and $\tilde{h}(k)$ is defined in terms of the t-transforms of the boundary values $\{\partial_x^j q(L,t)\}_0^{n-1}$ by an equation similar to (1.17), namely,

$$\tilde{h}(k) = \sum_{j=0}^{n-1} c_j(k)\tilde{h}_j(w(k)), \quad k \in \mathbb{C}. \tag{2.7}$$

Furthermore, the following global relation is valid:

$$\hat{q}_0(k) - \tilde{g}(k) + e^{-ikL}\tilde{h}(k) = e^{w(k)T}\int_0^L e^{-ikx}q(x,T)dx, \quad k \in \mathbb{C}. \tag{2.8}$$

Equation (2.6) is also valid if T in the definitions of $\tilde{g}(k)$ and $\tilde{h}(k)$ is replaced by t. This hold similarly for the global relation, where T in the RHS of (2.8) is replaced by t.

The canonical initial-boundary value problem for (1.1) can be solved by supplementing the global relation with the $n-1$ equations obtained through the transformations that leave $w(k)$ invariant and then solving these n equations for the following unknown functions:

$$\{\tilde{g}_j(k)\}_N^{n-1}, \quad \{\tilde{h}_j(k)\}_{n-N}^{n-1}.$$

Proof. Equation (43) and the application of Green's theorem either in the domain Ω_L or in the domain $\{0 < x < L, 0 < s < t\}$ yield either (2.8) or the equation obtained from (2.8) by replacing T with t. Solving the latter equation for the Fourier transform of $q(x,t)$ and then using the inverse Fourier transform we find an equation similar to (2.6), where the second and third integrals in the RHS of (2.6) involve integrals along the real line instead of integrals along ∂D^+ and ∂D^-. The deformation of the real line to ∂D^+ was discussed in the proof of Proposition 1.1. The justification for the deformation of the real line to ∂D^- is similar, where we note that the term $\exp[-(L-x)]$ is bounded in $\mathbb{C}^-$ and the change in the sign is due to the change of the orientation in ∂D^-.

The justification for replacing T with t is similar to that presented in the proof of Proposition 1.1. □

Remark 2.1. For T finite, $\tilde{g}(k)$ and $\tilde{h}(k)$ are entire functions; thus ∂D^+ and ∂D^- can be replaced by any other contours in $\mathbb{C}^+$ and $\mathbb{C}^-$ which approach ∂D^+ and ∂D^- as $k \to \infty$.

The Zeros of $\Delta(k)$

In the case of the canonical problem, the global relation and the $n-1$ equations obtained through the transformations $\{\nu_j(k)\}_0^{n-1}$, which leave $w(k)$ invariant, can be considered as a system of n equations for the n unknown t-transforms of the boundary values. The solution of this linear system involves the solution of the system obtained by neglecting the RHS of (2.8) plus an expression corresponding to the functions $\exp[w(k)T]\hat{q}_T(\nu_l)$, $l = 0, \ldots, n-1$, $\nu_0 = k$. The latter part involves $1/\Delta(k)$, where $\Delta(k)$ is the determinant of the above system. Since the function $\Delta(k)$, in addition to involving polynomials in k of degree up to $n-1$, also involves the exponential functions $\exp[-i\nu_l L]$, $l = 0, \ldots, n-1$, it is not a priori clear that the relevant exponentials are bounded as $k \to \infty$. It can be shown that for the canonical problem these exponentials are indeed bounded (see [9], [10], [11]). Actually, in the general case the relevant exponentials are bounded, provided that N boundary conditions are given at $x = 0$ and $n - N$ are given at $x = L$ (assuming of course that these boundary conditions are "independent," i.e., $\Delta(k)$ is not identically zero). Hence, if $\Delta(k)$ has *no* zeros in D, then the RHS of the global relation can be neglected (just as it happens with the canonical problem of the half-line). However, if $\Delta(k)$ *has* zeros in D, then the contours ∂D^+ and ∂D^- must be deformed to avoid these zeros, or alternatively the contribution of these zeros can be computed via a residue calculation (like the one performed for the Robin problem of the heat equation on the half-line).

The above discussion indicates that it is crucial to determine the zeros of $\Delta(k)$. Actually, because of Remark 2.1, it is sufficient to determine these zeros for large k. As $k \to \infty$, $\Delta(k)$ is asymptotically equal to a finite sum of exponential terms. $\Delta(k)$ is an entire function of finite order (the order of an entire function is a measure of its growth rate at infinity). For such functions there exists an extensive theory (see, for example, [79]) which implies that $\Delta(k)$ has infinitely many zeros accumulating at infinity which lie on specific rays in the complex k-plane. These rays can be determined as follows: Define the function $G(z)$ by

$$G(z) = e^z + a_1 e^{\lambda_1 z} + \cdots + a_n e^{\lambda_n z}, \tag{2.9}$$

where $\{a_j, \lambda_j\}^n$ are complex constants, such that the n-polygon with vertices at the points $\{1, \lambda_1, \ldots, \lambda_n\}$ is not degenerate. The zeros of $G(z)$ are clustered along the rays emanating from the origin with direction orthogonal to the sides of the polygon. Furthermore, these zeros can only accumulate at infinity on these rays regardless of the values of the complex constants $\{a_j\}_1^n$.

Example 2.1 (the heat equation). Let $q(x,t)$ satisfy the heat equation (1.20a) in the finite interval. In this case $w(k) = k^2$, ∂D^+ and ∂D^- are depicted in red and in green, respectively, in Figure 1.1, and $\tilde{g}(k)$ is defined by (1.20d). Thus the global relation (2.8) yields for all $k \in \mathbb{C}$

$$\hat{q}_0(k) - \left[ik\tilde{g}_0(k^2) + \tilde{g}_1(k^2)\right] + e^{-ikL}\left[ik\tilde{h}_0(k^2) + \tilde{h}_1(k^2)\right] = e^{k^2T}\hat{q}_T(k). \tag{2.10}$$

(i) *The Dirichlet Problem.* The solution is given by (2.6) with $w(k) = k^2$ and ∂D^+, ∂D^- as depicted in Figure 1.1, where

$$\tilde{g}(k) = \frac{1}{\Delta(k)}\left[-2ike^{-ikL}G_0(k^2) + 2ikH_0(k^2) + e^{ikL}\hat{q}_0(k) - e^{-ikL}\hat{q}_0(-k)\right], \tag{2.11a}$$

$$\tilde{h}(k) = \frac{1}{\Delta(k)}\left[-2ikG_0(k^2) + 2ike^{ikL}H_0(k^2) + \hat{q}_0(k) - \hat{q}_0(-k)\right], k \in D, \tag{2.11b}$$

$$\Delta(k) = e^{ikL} - e^{-ikL}, \quad k \in \mathbb{C}, \tag{2.11c}$$

and $G_0(k)$, $H_0(k)$ are the t-transforms of the Dirichlet boundary conditions; see (1.8), (2.3).

Indeed, the global relation (2.8) becomes

$$-\tilde{g}_1 + e^{-ikL}\tilde{h}_1 = e^{k^2T}\hat{q}_T(k) + N(k), \tag{2.12a}$$

where

$$N(k) = ikG_0(k^2) - ike^{-ikL}H_0(k^2) - \hat{q}_0(k).$$

Letting $k \to -k$ in (2.12a) we find

$$-\tilde{g}_1 + e^{ikL}\tilde{h}_1 = e^{k^2T}\hat{q}_T(-k) + N(-k). \tag{2.12b}$$

Equations (2.12) are two equations for the two unknown functions $\tilde{g}_1$ and $\tilde{h}_1$. The relevant determinant is given by (2.11c). The zeros of $\Delta(k)$ are on the real line, thus outside D, and hence we neglect the terms $\hat{q}_T(k)$ and $\hat{q}_T(-k)$ in (2.12). Solving the resulting equations for $\tilde{g}_1$ and $\tilde{h}_1$, we find

$$\tilde{h}_1 = \frac{1}{\Delta(k)}\left[N(-k) - N(k)\right], \quad \tilde{g}_1 = \frac{1}{\Delta(k)}\left[N(-k)e^{-ikL} - N(k)e^{ikL}\right].$$

Substituting these expressions in the definitions of $\tilde{g}(k)$ and $\tilde{h}(k)$, we find (2.11a) and (2.11b).

We note that the numerators of both $\tilde{g}(k)$ and $\tilde{h}(k)$ vanish at $k = 0$, and thus $k = 0$ is a removable singularity.

(ii) *The Robin Problem.* Let

$$q_x(0,t) - \gamma_1 q(0,t) = g_R(t), \quad q_x(L,t) + \gamma_2 q(L,t) = h_R(t), \quad 0 < t < T, \tag{2.13}$$

where γ_1 and γ_2 are positive constants, $\gamma_1 \neq \gamma_2$, and g_R, h_R are smooth functions compatible with $q_0(x)$ at $x = 0$ and $x = L$. The solution is given by (2.6) with $w(k) = k^2$ and ∂D^+, ∂D^- as depicted in Figure 1.1, where

$$\tilde{g}(k) = -\frac{1}{\Delta(k)}\big[(ik+\gamma_1)(ik+\gamma_2)e^{ikL}\hat{q}_0(k) + (ik+\gamma_1)(ik-\gamma_2)e^{-ikL}\hat{q}_0(-k) \\ -2ik(ik-\gamma_2)e^{-ikL}G_R(k^2) + 2ik(ik+\gamma_1)H_R(k^2)\big], \tag{2.14a}$$

$$\tilde{h}(k) = -\frac{1}{\Delta(k)}\big[(ik-\gamma_1)(ik-\gamma_2)\hat{q}_0(k) + (ik+\gamma_1)(ik-\gamma_2)\hat{q}_0(-k) \\ -2ik(ik-\gamma_2)G_R(k^2) + 2ik(ik+\gamma_1)e^{ikL}H_R(k^2)\big], \tag{2.14b}$$

$$\Delta(k) = (ik-\gamma_1)(ik-\gamma_2)e^{-ikL} - (ik+\gamma_1)(ik+\gamma_2)e^{ikL}, \tag{2.14c}$$

and $G_R(k)$, $H_R(k)$ are the t-transforms of the Robin data,

$$G_R(k) = \int_0^T e^{ks} g_R(s)ds, \quad H_R(k) = \int_0^T e^{ks} h_R(s)ds, \quad k \in \mathbb{C}. \tag{2.15}$$

Indeed, the t-transforms of the boundary conditions yield

$$\tilde{g}_1 = \gamma_1\tilde{g}_0 + G_R, \quad \tilde{h}_1 = -\gamma_2\tilde{h}_0 + H_R. \tag{2.16}$$

Hence the global relation (2.8) becomes

$$-(ik+\gamma_1)\tilde{g}_0 + (ik-\gamma_2)e^{-ikL}\tilde{h}_0 = e^{k^2T}\hat{q}_T(k) + N(k) \tag{2.17a}$$

with the known function $N(k)$ given by

$$N(k) = G_R(k^2) - e^{-ikL}H_R(k^2) - \hat{q}_0(k).$$

Replacing k by $-k$ in (2.17a), we find

$$(ik-\gamma_1)\tilde{g}_0 - (ik+\gamma_2)e^{ikL}\tilde{h}_0 = e^{k^2T}\hat{q}_T(-k) + N(-k). \tag{2.17b}$$

The relevant determinant associated with (2.17) is given by (2.14c). The zeros of $\Delta(k)$ are on the real axis; thus again we neglect the terms $\hat{q}_T(k)$, $\hat{q}_T(-k)$ and solve the resulting equations for $\tilde{g}_0$ and $\tilde{h}_0$,

$$\tilde{g}_0 = \frac{1}{\Delta(k)}\left[(ik+\gamma_2)e^{ikL}N(k) + (ik-\gamma_2)e^{-ikL}N(-k)\right],$$

$$\tilde{h}_0 = \frac{1}{\Delta(k)}\left[(ik-\gamma_1)N(k) + (ik+\gamma_1)N(-k)\right].$$

Substituting these expressions in the definition of $\tilde{g}(k)$, $\tilde{h}(k)$ and using (2.16), we find (2.14a) and (2.14b).

Example 2.2 (the first Stokes equation). Let $q(x,t)$ satisfy the canonical problem of the first Stokes equation (1.24a), i.e.,

$$q(0,t) = g_0(t), \quad q(L,t) = h_0(t), \quad q_x(L,t) = h_1(t), \quad 0 < t < T. \tag{2.18}$$

The solution is given by (2.6) with $w(k) = ik - ik^3$ and with ∂D^+, ∂D^- depicted in red and in green, respectively, in Figure 1.5, where $\tilde{g}(k)$, $\tilde{h}(k)$ are defined in terms of ν_1, ν_2, and in terms of G_0, H_0, H_1 (the t-transform of the boundary conditions) by the following expressions (ν_1, ν_2 are defined in (1.52)):

$$\begin{aligned}\tilde{g}(k) = \frac{1}{\Delta(k)}\Big\{&\left(\nu_1 e^{-i\nu_1 L} - \nu_2 e^{-i\nu_2 L}\right)\hat{q}_0(k) + e^{-ikL}\left(\nu_2\hat{q}_0(\nu_2) - \nu_1\hat{q}_0(\nu_1)\right)\\ &+(k^2-1)(\nu_2-\nu_1)e^{-ikL}G_0(w(k))\\ &+e^{-ikL}\left[\nu_1(k^2-\nu_1^2)e^{-i\nu_1 L} - \nu_2(k^2-\nu_2^2)e^{-i\nu_2 L}\right]H_0(w(k))\\ &+ie^{-ikL}\left[\nu_1(\nu_1-k)e^{-i\nu_1 L} - \nu_2(\nu_2-k)e^{-i\nu_2 L}\right]H_1(w(k))\Big\},\end{aligned} \tag{2.19a}$$

$$\begin{aligned}\tilde{h}(k) = \frac{1}{\Delta(k)}\Big\{&(\nu_1-\nu_2)\hat{q}_0(k) - \nu_1\hat{q}_0(\nu_1) + \nu_2\hat{q}_0(\nu_2) + (k^2-1)(\nu_2-\nu_1)G_0(w(k))\\ &+\left[\nu_1(k^2-\nu_1^2)e^{-i\nu_1 L} - \nu_2(k^2-\nu_2^2)e^{-i\nu_2 L}\right]H_0(w(k))\\ &+i\left[\nu_1(\nu_1-k)e^{-i\nu_1 L} - \nu_2(\nu_2-k)e^{-i\nu_2 L}\right]H_1(w(k))\Big\},\end{aligned} \tag{2.19b}$$

$$\Delta(k) = (\nu_2-\nu_1)e^{-ikL} + \nu_1 e^{-i\nu_1 L} - \nu_2 e^{-i\nu_2 L}. \tag{2.19c}$$

Indeed, we supplement the global relation (2.8) with the two equations obtained from (2.8) by replacing k with ν_1 and ν_2 which are defined in (1.52). It will turn out (see below) that the relevant determinant $\Delta(k)$ is given by (2.19c). This function does *not* have zeros in D for large k: As $k \to \infty$, $\Delta(k)$ is proportional to $\Delta_\infty(k)$,

$$\Delta_\infty(k) = (\alpha^2-\alpha)e^{-ikL} + \alpha e^{-i\alpha kL} - \alpha^2 e^{-i\alpha^2 kL}, \quad \alpha = e^{\frac{2i\pi}{3}}.$$

Letting $z = -ik$ in this equation and comparing the resulting equation with the expression $G(z)$ defined in (2.9), it follows that $\lambda_1 = \alpha$ and $\lambda_2 = \alpha^2$. Hence the relevant rays in the complex k-plane are shown in Figure 2.1(a). Using the transformation $z = -ik$, we conclude that as $k \to \infty$ the zeros of $\Delta(k)$ are on the rays shown in Figure 2.1(b).

Thus, we neglect the term $\hat{q}_T(k)$ and we obtain the following three equations for the three unknown functions $\tilde{g}_1$, $\tilde{g}_2$, $\tilde{h}_2$:

$$\begin{aligned} ik\tilde{g}_1 + \tilde{g}_2 - e^{-ikL}\tilde{h}_2 &= N(k),\\ i\nu_1\tilde{g}_1 + \tilde{g}_2 - e^{-i\nu_1 L}\tilde{h}_2 &= N(\nu_1),\\ i\nu_2\tilde{g}_1 + \tilde{g}_2 - e^{-i\nu_2 L}\tilde{h}_2 &= N(\nu_2)\end{aligned}$$

with

$$N(k) = (k^2-1)\left(G_0 - e^{-ikL}H_0\right) + ike^{-ikL}H_1 - \hat{q}_0(k).$$

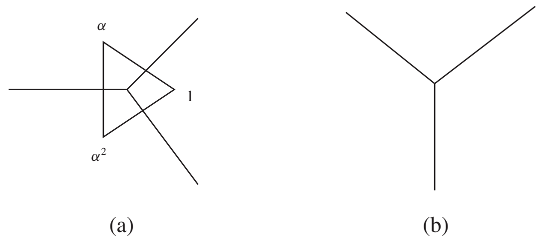


Figure 2.1. (a) *The zeros in the complex z-plane.* (b) *The zeros in the complex k-plane.*

Solving this linear system we find the following expressions for $i\tilde{g}_1$, $\tilde{g}_2$, $\tilde{h}_2$:

$$\begin{aligned} i\tilde{g}_1 &= \frac{1}{\Delta(k)}\Big\{[N(\nu_2) - N(\nu_1)]e^{-ikL} + [N(k) - N(\nu_2)]e^{-i\nu_1 L} \\ &\qquad + [N(\nu_1) - N(k)]e^{-i\nu_2 L}\Big\}, \\ \tilde{g}_2 &= \frac{1}{\Delta(k)}\Big\{[\nu_2 N(\nu_1) - \nu_1 N(\nu_2)]e^{-ikL} + [kN(\nu_2) - \nu_2 N(k)]e^{-i\nu_1 L} \\ &\qquad + [\nu_1 N(k) - kN(\nu_1)]e^{-i\nu_2 L}\Big\}, \\ \tilde{h}_2 &= \frac{1}{\Delta(k)}\Big[(\nu_1 - \nu_2)N(k) - \nu_1 N(\nu_1) + \nu_2 N(\nu_2)\Big]. \end{aligned}$$

Substituting these expressions in the definitions of $\tilde{g}(k)$ and $\tilde{h}(k)$, we find (2.19a) and (2.19b). To simplify these expressions we have used the following identities:

$$\nu_1 + \nu_2 = k, \quad \nu_1\nu_2 = k^2 - 1, \quad \nu_1(\nu_1^2 - 1) = \nu_2(\nu_2^2 - 1),$$

$$\nu_1(\nu_1 - k) = -\nu_1\nu_2, \quad \nu_2(\nu_2 - k) = -\nu_1\nu_2, \quad \nu_1(k^2 - \nu_1^2) = \nu_1\nu_2(k + \nu_1),$$

$$\nu_2(k^2 - \nu_2^2) = \nu_1\nu_2(k + \nu_2).$$

2.1 The Classical Representations: Return to the Real Line

This situation is similar with that discussed in section 1.1. Consider for example the heat equation. Replacing T by t in the global relation (2.10), and also replacing k by $-k$, we find the following equations, which are valid for all $k \in \mathbb{C}$:

$$\begin{aligned} &\hat{q}_0(k) - \int_0^t e^{k^2 s}[ikq(0,s) + q_x(0,s)]ds \\ &\quad + e^{-ikL}\int_0^t e^{k^2 s}[ikq(L,s) + q_x(L,s)]ds = e^{k^2 t}\hat{q}(k,t), \end{aligned} \tag{2.20a}$$

$$\begin{aligned} &\hat{q}_0(-k) - \int_0^t e^{k^2 s}[-ikq(0,s) + q_x(0,s)]ds \\ &\quad + e^{ikL}\int_0^t e^{k^2 s}[-ikq(L,s) + q_x(L,s)]ds = e^{k^2 t}\hat{q}(-k,t), \end{aligned} \tag{2.20b}$$

where $\hat{q}_0(k)$ and $\hat{q}(k,t)$ denote the Fourier transform of $q_0(x)$ and $q(x,t)$.

(i) *The Dirichlet Problem.* In this case we must eliminate $q_x(0,s)$ and $q_x(L,s)$ from (2.20). In order to eliminate $q_x(0,s)$ we subtract them:

$$\hat{q}_0(-k) - \hat{q}_0(k) + 2ikG_0(k^2,t) - ik\left(e^{ikL} + e^{-ikL}\right)H_0(k^2,t)$$
$$+\Delta(k)\int_0^t e^{k^2 s} q_x(L,s)ds = e^{k^2 t}[\hat{q}(-k,t) - \hat{q}(k,t)], \quad k \in \mathbb{C}, \tag{2.21}$$

where $\Delta(k)$ is as defined in (2.11c). In order to eliminate $q_x(L,s)$ we evaluate (2.21) at those values of k at which $\Delta(k) = 0$, i.e., $k = n\pi/L$, $n \in \mathbb{Z}$. Then (2.21) yields the following expression for the finite sine transform of $q(x,t)$:

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right) q(x,t)dx$$
$$= e^{-\frac{n^2\pi^2}{L^2}t}\left\{\hat{q}_n^{(s)} + k\left[G_0\left(\frac{n^2\pi^2}{L^2},t\right) - e^{in\pi}H_0\left(\frac{n^2\pi^2}{L^2},t\right)\right]\right\}, \tag{2.22a}$$

where $\hat{q}_n^{(s)}$ denotes the sine transform of $q_0(x)$ evaluated at $k = n\pi/L$,

$$\hat{q}_n^{(s)} = \int_0^L \sin\left(\frac{n\pi x}{L}\right) q_0(x)dx, \quad n \in \mathbb{Z}. \tag{2.22b}$$

Inverting the finite sine transform of $q(x,t)$ appearing in the LHS of (2.22a), we find the classical sine-series representation,

$$q(x,t) = \frac{2}{L}\sum_{n=1}^{\infty} e^{\frac{-n^2\pi^2}{L^2}t}\sin\left(\frac{n\pi x}{L}\right)$$
$$\times\left\{\hat{q}_n^{(s)} + k\left[G_0\left(\frac{n^2\pi^2}{L^2},t\right) - e^{in\pi}H_0\left(\frac{n^2\pi^2}{L^2},t\right)\right]\right\}. \tag{2.22c}$$

(ii) *The Robin Problem.* Let $q(x,t)$ satisfy the boundary conditions (2.13). In this case, replacing in (2.20) $q_x(0,s)$ and $q_x(L,s)$ by $q(0,s) + \gamma_1 g_R(s)$ and $q(L,s) - \gamma_2 h_R(s)$, respectively, we find

$$\hat{q}_0(k) - (ik+\gamma_1)\int_0^t e^{k^2 s}q(0,s)ds + (ik-\gamma_2)e^{-ikL}\int_0^t e^{k^2 s}q(L,s)ds$$
$$-G_R(k^2,t) + e^{-ikL}H_R(k^2,t) = e^{k^2 t}\hat{q}(k,t), \tag{2.23a}$$

$$\hat{q}_0(-k) - (-ik+\gamma_1)\int_0^t e^{k^2 s}q(0,s)ds - (ik+\gamma_2)e^{ikL}\int_0^t e^{k^2 s}q(L,s)ds$$
$$-G_R(k^2,t) + e^{ikL}H_R(k^2,t) = e^{k^2 t}\hat{q}(-k,t), \tag{2.23b}$$

where $G_R(k,t)$ and $H_R(k,t)$ denote the t-transforms from 0 to t of the given boundary conditions,

$$G_R(k,t) = \int_0^t e^{ks}g_R(s)ds, \quad H_R(k,t) = \int_0^t e^{ks}h_R(s)ds, \quad k \in \mathbb{C}. \tag{2.24}$$

In order to eliminate the term involving $q(0,s)$ from (2.23) we multiply (2.23a) and (2.23b) by $(ik-\gamma_1)$ and $(ik+\gamma_1)$, respectively, and add the resulting expressions. This yields

$$\begin{aligned}&(ik-\gamma_1)\hat{q}_0(k)+(ik+\gamma_1)\hat{q}_0(-k)+\Delta(k)\int_0^t e^{k^2s}q(L,s)ds\\&-2ikG_R(k^2,t)+\left[(ik-\gamma_1)e^{-ikL}+(ik+\gamma_1)e^{ikL}\right]H_R(k^2,t)\\&=e^{k^2t}\left[(ik-\gamma_1)\hat{q}(k,t)+(ik+\gamma_1)\hat{q}(-k,t)\right],\end{aligned}\tag{2.25}$$

where $\Delta(k)$ is defined by (2.14c). In order to eliminate the term involving $q(L,s)$ we choose those values of k denoted by k_n for which $\Delta(k)=0$, i.e.,

$$(ik_n-\gamma_1)(ik_n-\gamma_2)e^{-ik_nL}=(ik_n+\gamma_1)(ik_n+\gamma_2)e^{ik_nL}.\tag{2.26a}$$

Simplifying this equation we find

$$\frac{\tan(k_nL)}{k_n}=\frac{\gamma_1+\gamma_2}{k_n^2-\gamma_1\gamma_2}\text{ and }k_0=0.\tag{2.26b}$$

Evaluating (2.25) at $k=k_n$ we find

$$\begin{aligned}&\int_0^L\left[e^{ik_nx}+\frac{k_n+i\gamma_1}{k_n-i\gamma_1}e^{-ik_nx}\right]q(x,t)dx\\&=e^{-k_n^2t}\left[\hat{q}_n^{(R)}-\frac{2k_n}{k_n-i\gamma_1}G_R(k_n^2,t)+\left(e^{ik_nL}+\frac{k_n+i\gamma_1}{k_n-i\gamma_1}e^{-ik_nL}\right)H_R(k_n^2,t)\right],\end{aligned}\tag{2.27a}$$

where $\hat{q}_n^{(R)}$ is defined by

$$\hat{q}_n^{(R)}=\int_0^L\left[e^{ik_nx}+\frac{k_n+i\gamma_1}{k_n-i\gamma_1}e^{-ik_nx}\right]q_0(x)dx\tag{2.27b}$$

with k_n defined by (2.26b).

As it was noted in section 1.1, the above approach of deriving classical representations has the advantage of avoiding the determination of the "proper" transform (as well as avoiding integration by parts). On the other hand, it requires the knowledge of how to invert the integral transforms of $q(x,t)$. It is possible to obtain the classical representations *without* using the Sturm–Liouville theory to construct these transforms by starting with the novel integral representations and then using contour deformation. As an example we consider the heat equation.

(i) *The Dirichlet Problem.* The novel integral representation is given by (2.6) with $w=k^2$, with the contours ∂D^+, ∂D^- as depicted in Figure 1.1, and with the functions $\tilde{g}(k)$ and $\tilde{h}(k)$ defined by (2.11). We will use Cauchy's theorem in the domains $E^+=\mathbb{C}\backslash D^+$ and $E^-=\mathbb{C}\backslash D^-$ (which are the domains between the real axis and ∂D^+, ∂D^-, respectively) to compute the last two integrals in the RHS of (2.6). In this respect we note the following:

(a) We can replace T with t in the expressions for $\tilde{g}(k)$ and $\tilde{h}(k)$ and we will refer to the resulting expressions as $\tilde{g}(k,t)$ and $\tilde{h}(k,t)$, respectively.

(b) Cauchy's theorem in E^+ and E^- implies

$$\int_{\partial D^+} = \fint_{-\infty}^{\infty} - i\pi \sum. \quad \int_{\partial D^-} = -\fint_{-\infty}^{\infty} - i\pi \sum,$$

where $\fint$ denotes the principal value integral and $\sum$ denotes the sum of the residues. The functions $\tilde{g}$ and $\exp[-ikL]\tilde{h}$ which appear in the integrands of the respective integrals differ only in the terms involving $\hat{q}_0(k)$, and thus the contribution of the principal value integrals is given by

$$-\frac{1}{2\pi}\fint_{-\infty}^{\infty} \frac{1}{\Delta(k)} e^{ikx-k^2t}[e^{ikL}\hat{q}_0(k) - e^{-ikL}\hat{q}_0(k)]dk$$

which cancels the first term on the RHS of (2.8).

(c) The functions $\tilde{g}$ and $\exp[-ikL]\tilde{h}$ differ only by the coefficient $\exp[2ikL]$ of $\hat{q}_0(k)$, and thus these two functions yield the same residues at the poles $k = n\pi/L$. Hence,

$$q(x,t) = \frac{2i\pi}{2\pi} \sum_{n\in Z} \left. \frac{e^{ikx-k^2t}[\Delta(k)\tilde{g}(k,t)]}{iL(e^{ikL} + e^{-ikL})} \right|_{k=n\pi/L}.$$

Splitting this sum into a sum over $n \in \mathbb{Z}^-$ and a sum over $n \in \mathbb{Z}^+$ and letting $n \to -n$ in the former sum, we find (2.22c).

(ii) *The Robin Problem.* In this case the functions $\tilde{g}(k)$ and $\tilde{h}(k)$ are defined by (2.19a) and (2.19b). Again we observe that the functions $\tilde{g}$ and $\exp[-ikL]\tilde{h}$ differ only in the terms involving $\hat{q}_0(k)$, and thus the contribution of the principal value integral is given by

$$-\frac{1}{2\pi}\fint_{-\infty}^{\infty} \frac{1}{\Delta(k)} e^{ikx-k^2t}[(ik-\gamma_1)(ik-\gamma_2)e^{-ikL}\hat{q}_0(k) - (ik+\gamma_1)(ik+\gamma_2)e^{ikL}\hat{q}_0(k)]dk$$

which cancels the first term on the RHS of (2.6). Furthermore, the coefficients of the terms involving $\hat{q}_0(k)$ in $\tilde{g}$ and in $\exp[-ikL]\tilde{h}$ are equal at $k = k_n$. Thus

$$q(x,t) = i\sum_{k_n} \left. \frac{1}{\Delta'(k)} e^{ikx-k^2t}\Delta(k)\tilde{g}(k,t) \right|_{k=k_n},$$

where k_n solves (2.26) and Δ' denotes the derivative of $\Delta(k)$ with respect to k, whereas $\tilde{g}$ and Δ are defined by (2.14a) and (2.14c). The transformation $k_n \to -k_n$ leaves (2.26b) invariant, and therefore by splitting the above sum into a sum over k_n positive and over k_n negative and letting $k_n \to -k_n$ in the former sum we find that

$$\begin{aligned} q(x,t) = -i\sum_{n=1}^{\infty} \frac{e^{-k_n^2t}}{\Delta'(k_n)} \Big(& 2(ik_n+\gamma_2)e^{ik_nL}[ik_n\cos(k_nx) + i\gamma_1\sin(k_nx)]\hat{q}_0(k_n) \\ & +2(ik_n-\gamma_2)e^{-ik_nL}[ik_n\cos(k_nx) + i\gamma_1\sin(k_nx)]\hat{q}_0(-k_n) \\ & +4k_n\left\{[k_n\cos(k_nL) + \gamma_2\sin(k_nL)]\cos(k_nx)\right\}G_R\left(k_n^2\right) \\ & +4k_n\left\{[k_n\sin(k_nL) - \gamma_2\cos(k_nL)]\sin(k_nx)\right\}G_R\left(k_n^2\right) \\ & -4k_n[k_n\cos(k_nx) + \gamma_1\sin(k_nx)]H_R\left(k_n^2\right)\Big), \; \gamma_1 > 0, \gamma_2 > 0. \end{aligned} \tag{2.28}$$

In the case that $\gamma_1 = \gamma_2 = 0$, the RHS of (2.28) contains the additional term $\hat{q}_0(0)/2L$.

2.2 Forced Problems

Let $q(x,t)$ satisfy the inhomogeneous equation (1.65) in the domain Ω_L where $f(x,t)$ has sufficient smoothness. Let $q(x,t)$ satisfy the initial condition $q_0(x)$ and appropriate boundary conditions at $x=0$ and $x=L$.

Proceeding as with the case of the forced problem on the half-line we find that $\hat{q}_0(k)$ must now be replaced by $\hat{q}_0(k)+F_L(k,t)$, where F_L is defined by the equation

$$F_L(k,t)=\int_0^t\left(\int_0^L e^{-ik\xi+w(k)\tau}f(\xi,\tau)d\xi\right)d\tau. \tag{2.29}$$

2.3 Green's Function Type Representations

Following steps similar to those used in section 1.3, it is always possible to rewrite the solution in the form

$$\begin{aligned} q(x,t) = &\int_0^L G^{(I)}(x,t,\xi)q_0(\xi)d\xi+\sum_{j=0}^{N-1}\int_0^t G_j^{(B)}(x,t,s)g_j(s)ds \\ &+\sum_{j=0}^{n-N-1} H_j^{(B)}(x,t,s)h_j(s)ds. \end{aligned}$$

For brevity of presentation we consider only the case of homogeneous boundary conditions.

2.3.1. The Heat Equation

Let $q(x,t)$ satisfy the heat equation with homogeneous Dirichlet boundary conditions. Then (2.11a) and (2.11b) with $g_0=h_0=0$ yield

$$\tilde{g}=\frac{1}{\Delta(k)}\left[e^{ikL}\hat{q}_0(k)-e^{-ikL}\hat{q}_0(-k)\right],$$

$$\tilde{h}=\frac{1}{\Delta(k)}\left[\hat{q}_0(k)-\hat{q}_0(-k)\right],\quad k\in\mathbb{C},$$

where $\Delta(k)$ is defined by (2.11c) and $\hat{q}_0(k)$ denotes the Fourier transform of the initial condition $q_0(x)$. Thus, (2.6) yields

$$\begin{aligned} q(x,t) = &\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ikx-k^2t}\hat{q}_0(k)dk-\frac{1}{2\pi}\int_{\partial D^+}\frac{1}{\Delta(k)}e^{ikx-k^2t}[e^{ikL}\hat{q}_0(k)-e^{-ikL}\hat{q}_0(-k)] \\ &-\frac{1}{2\pi}\int_{\partial D^-}\frac{1}{\Delta(k)}e^{ikx-k^2t}\left[e^{-ikL}\hat{q}_0(k)-e^{-ikL}\hat{q}_0(-k)\right],\quad (x,t)\in\Omega_L, \end{aligned} \tag{2.30}$$

where ∂D^+ and ∂D^- are the red and green curves, respectively, depicted in Figure 1.1.

Furthermore, for the case of homogeneous Dirichlet boundary conditions, the classical sine-series representation (2.22c) becomes

$$q(x,t)=\frac{2}{L}\sum_{n=1}^{\infty}e^{-\frac{n^2\pi^2}{L^2}t}\sin\left(\frac{n\pi x}{L}\right)\hat{q}_n^{(s)}. \tag{2.31}$$

Equation (2.30) can be rewritten in the form

$$q(x,t) = \int_0^L G^{(I)}(x,\xi,t)q_0(\xi)d\xi, \quad (x,t) \in \Omega_L, \tag{2.32}$$

where $G^{(I)}$ is given by

$$\begin{aligned} G^{(I)}(x,\xi,t) = {} & \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-\xi)-k^2t}dk \\ & -\frac{1}{2\pi}\int_{\partial D_0^+} \frac{e^{-k^2t}}{\Delta(k)}\left[e^{ik(x-\xi)+ikL} - e^{ik(x+\xi)-ikL}\right]dk \\ & -\frac{1}{2\pi}\int_{\partial D_0^-} \frac{e^{-k^2t-ikL}}{\Delta(k)}\left[e^{ik(x-\xi)} - e^{ik(x+\xi)}\right]dk \end{aligned} \tag{2.33}$$

and ∂D_0^+, ∂D_0^- denote the curves obtained by deforming ∂D^+ and ∂D^- to pass above and below $k = 0$, respectively. The reason for replacing $\partial D^\pm$ by $\partial D_0^\pm$ (which is allowed due to analyticity considerations) is that in what follows we will split the integrants of the second and third integrals in the RHS of (2.33) and we want to avoid the pole at $k = 0$. The first term in the RHS of (2.33) equals the first term in the RHS of (1.68a) which equals $1/2\sqrt{\pi t}$; however, the second and third terms in the RHS of (2.33) *cannot* be computed in a closed form. Hence $G^{(I)}$ for the finite interval, in contrast to the case of the half-line, cannot be found explicitly.

Using the representation (2.32), it is straightforward to compute the *trace* of $G^{(I)}$, which is defined by the equation

$$K(t) = \int_0^L G^{(I)}(x,x,t)dx, \quad t > 0. \tag{2.34}$$

Indeed, letting $\xi = x$ in (2.33) we find

$$\begin{aligned} G(x,x,t) = {} & \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-k^2t}dk - \frac{1}{2\pi}\int_{\partial D_0^+}\frac{1}{\Delta(k)}e^{-k^2t+ikL}dk - \frac{1}{2\pi}\int_{\partial D_0^-}\frac{1}{\Delta(k)}e^{-k^2t-ikL}dk \\ & + \frac{1}{2\pi}\int_{\partial D_0^+}\frac{e^{-k^2t+2ikx}dk}{e^{2ikL}-1} + \frac{1}{2\pi}\int_{\partial D_0^+}\frac{e^{-k^2t+2ikx}dk}{e^{2ikL}-1}. \end{aligned} \tag{2.35}$$

The first integral on the RHS of (2.35) can be computed explicitly, the second and the third integrals are equal, and after integrating with respect to x, the fourth and fifth integrals yield equal contributions which can be computed explicitly. Hence,

$$K(t) = \frac{L}{2\sqrt{\pi t}} - \frac{1}{2} - \frac{L}{\pi}\int_{\partial D_0^+}\frac{e^{-k^2t}}{1-e^{-2ikL}}dk, \tag{2.36}$$

where we have used

$$\int_{-\infty}^{\infty} e^{-l^2}dl = \sqrt{\pi}, \qquad \int_{\partial D_0^-} e^{-l^2}\frac{dl}{l} = -i\pi.$$

Equation (2.31) can also be rewritten in the form (2.32), where

$$G^{(I)}(x,\xi,t) = \frac{2}{L}\sum_{1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi \xi}{L}\right) e^{-\frac{n^2\pi^2}{L^2}t}, \quad (x,t) \in \Omega_L.$$

Hence, $K(t)$ also satisfies

$$K(t) = \sum_{n=1}^{\infty} e^{-\frac{n^2\pi^2}{L^2}}. \tag{2.37}$$

2.3.2. The First Stokes Equation

Let $q(x,t)$ satisfy the canonical problem of the first Stokes equation with homogeneous boundary conditions; see (2.18) with $g_0 = h_0 = h_1 = 0$. Then (2.19a) and (2.19b) become

$$\tilde{g}(k) = \frac{1}{\Delta(k)}\left[\left(\nu_1 e^{-i\nu_1 L} - \nu_2 e^{-i\nu_2 L}\right)\hat{q}_0(k) + e^{-ikL}\left(\nu_2 \hat{q}_0(\nu_2) - \nu_1 \hat{q}_0(\nu_1)\right)\right],$$

$$\tilde{h}(k) = \frac{1}{\Delta(k)}\left[(\nu_1 - \nu_2)\hat{q}_0(k) - \nu_1 \hat{q}_0(\nu_1) + \nu_2 \hat{q}_0(\nu_2)\right],$$

where ν_1, ν_2 are defined by (1.52) and $\Delta(k)$ is defined by (2.19c).

Substituting the above expression in (2.6) we find that $q(x,t)$ can be expressed in the form (2.32), where $G^{(I)}$ is given by

$$\begin{aligned} G^{(I)}(x,\xi,t) = {} & \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-\xi)-w(k)t}dk - \frac{1}{2\pi}\int_{\partial D^+} \frac{1}{\Delta(k)} e^{ikx-w(k)t} \\ & \times \left[\left(\nu_1 e^{-i\nu_1 L} - \nu_2 e^{-i\nu_2 L}\right)e^{-ik\xi} + e^{-ikL}\left(\nu_2 e^{-i\nu_2\xi} - \nu_1 e^{-i\nu_1\xi}\right)\right]dk \\ & - \frac{1}{2\pi}\int_{\partial D^-} \frac{1}{\Delta(k)} e^{ik(x-L)-w(k)t}\left[(\nu_1 - \nu_2)e^{-ik\xi} - \nu_1 e^{-i\nu_1\xi} + \nu_2 e^{-i\nu_2\xi}\right]dk, \end{aligned} \tag{2.38}$$

where $w(k) = i(k - k^3)$ and the contour ∂D^+, ∂D^- are as depicted in Figure 1.5.

Letting $x = \xi$ in the above equation and then integrating the resulting expression with respect to x from 0 to L, we find that the trace K is given by

$$\begin{aligned} K(t) = {} & \frac{L}{2\pi}\left[\int_{-\infty}^{\infty} e^{-w(k)t}dk - \int_{\partial D^+} \frac{1}{\Delta(k)} e^{-w(k)t}(\nu_1 e^{-i\nu_1 L} - \nu_2 e^{-i\nu_2 L})dk \right. \\ & \left. - \int_{\partial D^-} \frac{1}{\Delta(k)} e^{-w(k)t-ikL}(\nu_1 - \nu_2)dk\right] \\ & + \frac{i}{2\pi}\left(\int_{\partial D^+} + \int_{\partial D^-}\right)\frac{1}{\Delta(k)} e^{-w(k)t-kL}\left[\frac{\nu_1}{\nu_2}\left(e^{i(k-\nu_1)L} - 1\right) - \frac{\nu_2}{\nu_1}\left(e^{i(k-\nu_2)L} - 1\right)\right]dk. \end{aligned} \tag{2.39}$$

Chapter 3

Asymptotics and a Novel Numerical Technique

The integral representations obtained by the new method are formulated in the complex k-plane. Hence, it is possible to study the asymptotic properties of the solution by employing the well-developed techniques of complex analysis for the asymptotic evaluation of integrals.

Consider for example the limit of the trace $K(t)$ of the Green's function associated with the Dirichlet problem of the heat equation on the finite interval. This function is given by (2.36), which immediately yields the well-known formula

$$K(t) = \frac{L}{2\sqrt{\pi t}} - \frac{1}{2} + O(t^{\infty}). \tag{3.1}$$

Similarly, it is straightforward to compute the limit of the analogous function for the first Stokes equation; see (2.39).

By employing either the stationary phase or the steepest descent method, it is also straightforward to compute the long-time asymptotics of the solutions. For example, the following result is derived in [13].

Let $q(x,t)$ satisfy

$$q_t = q_{xxx} + \beta q_x, \quad 0 < x < \infty, \quad 0 < t < \infty, \tag{3.2}$$

$\beta > 0$ constant, with the following initial and boundary conditions:

$$q(x,0) = q_0(x) \in H^2(\mathbb{R}^+), \quad q(0,t) = g_0(t) \in H^1(\mathbb{R}^+), \quad q_x(0,t) = g_1(t) \in H^{\frac{2}{3}}(\mathbb{R}^+). \tag{3.3}$$

Then $q(x,t)$, $x = vt$, v positive and of O(1), satisfies

$$\begin{aligned} q(vt,t) = {} & \frac{1}{\sqrt{12\pi\gamma t}}[(P(\gamma) - P(\nu(\gamma)))e^{i\phi} + (P(-\gamma) - P(\nu(-\gamma)))e^{-i\phi}] \\ & + O\left(t^{-\frac{3}{2}}\right), \quad t \to \infty, \end{aligned} \tag{3.4}$$

where

$$\gamma = \sqrt{\frac{v+\beta}{3}}, \tag{3.5a}$$

$$\phi(t) = 2\gamma^3 t - \frac{\pi}{4}, \tag{3.5b}$$

$$\nu(k) = -\frac{1}{2}\left(k + i\sqrt{3k^2 - 4\beta}\right), \tag{3.5c}$$

$$P(k) = \hat{q}_0(k) + (k^2 - \beta)G_0^\infty(k) - ikG_1^\infty(k), \tag{3.5d}$$

$\hat{q}_0(k)$ is defined by (1.6), and

$$G_j^\infty(k) = \int_0^\infty e^{i(k^3-\beta k)s} g_j(s)ds, \quad j = 0, 1, \quad \text{Im}\,(k^3 - \beta k) \geq 0. \tag{3.5e}$$

Similar results can be obtained for other initial-boundary value problems; see [13], [31].

Flyer and the author [3], starting with the integral representations presented in Chapters 1 and 2, have introduced a new method for the numerical evaluation of the solution. This is based on the fact that it is possible, using simple contour deformations in the complex k-plane, to obtain integrals involving integrands with a strong decay for large k.

In order to present this new numerical approach in its simplest form, we will only consider initial and boundary conditions for which the associated Fourier and t-transforms can be computed analytically. In this case, the numerical implementation consists of the following.

1. Perform simple contour deformations in the complex-k plane such that the deformed integration paths are in regions where the integrands decay exponentially for large k. This yields rapid convergence of the numerical scheme.

2. For algorithmic convenience and simplicity, make a change of variables which maps the contours from the complex plane to the real line.

3.1 The Heat Equation on the Half-Line

Recall that the solution of the Dirichlet problem of the heat equation on the half-line is given by (16).

Let us first consider the integral whose contour runs along ∂D^+. Recalling the definition of G_0, we find

$$e^{ikx-k^2t}G_0(k^2, t) = e^{ikx}\int_0^t e^{-k^2(t-s)}g_0(s)ds. \tag{3.6}$$

Since $\exp[ikx]$ is bounded and analytic for Im $k > 0$, and $e^{-k^2(t-s)}$ with $t \geq s$ is bounded and analytic for Re $(k^2) > 0$, it follows that for this term, the contour ∂D^+ can be deformed to any contour $\mathcal{L}$ in the unshaded domain of the upper half complex k-plane; see Figure 3 of the introduction. The term $e^{ikx-k^2t}\hat{q}_0(-k)$ involves the factors e^{ikx} and $\hat{q}_0(-k)$, which are bounded and analytic for Im $k > 0$, and the factor e^{-k^2t}, which is bounded and analytic for Re $k^2 > 0$. Hence, the contour ∂D^+ for this term can also be deformed to $\mathcal{L}$.

Splitting $\hat{q}_0(k)$ into two terms and then substituting them into the integral along the real axis results in the following integrand:

$$e^{-k^2t}\left[\int_0^x e^{ik(x-\xi)}q_0(\xi)d\xi + \int_x^\infty e^{ik(x-\xi)}q_0(\xi)d\xi\right]. \tag{3.7}$$

In the first term, $x - \xi \geq 0$, thus this term can be deformed to $\mathcal{L}$. However, for the second term in (3.7), $x - \xi \leq 0$, thus for this term the contour along the real axis can be deformed to a contour $\mathcal{L}^-$ in the unshaded domain of the lower half complex k-plane.

For the integral along the real line, the term $\hat{q}_0(k)$ is in general analytic only for $\operatorname{Im} k < 0$. However, depending on the properties of $q_0(x)$ it is sometimes possible to extend the domain of analyticity to the upper half of the complex k-plane. For example, this is possible if $q_0(x)$ is such that $q_0(x)e^{\alpha x}$, $\alpha > 0$, is square integrable on $[0, \infty)$. If $q_0(x)$ belongs to this restricted class, then the contour along the real axis can also be deformed to the contour $\mathcal{L}$.

Example 3.1 Let

$$q_0(x) = xe^{-a^2 x}, \quad 0 < x < \infty, \quad g_0(t) = \sin bt, \quad t > 0, \tag{3.8}$$

where a and b are real numbers. Then

$$\hat{q}_0(k) = \frac{1}{(ik + a^2)^2}, \quad G_0(k, t) = \frac{1}{2i}\left[\frac{e^{(k+ib)t} - 1}{k + ib} - \frac{e^{(k-ib)t} - 1}{k - ib}\right]. \tag{3.9}$$

Hence, (16) yields

$$\begin{aligned} q(x, t) = \frac{1}{2\pi}\int_{\mathcal{L}} \Bigg\{ & e^{ikx - k^2 t}\left[\frac{1}{(ik + a^2)^2} - \frac{1}{(-ik + a^2)^2}\right] \\ & -ke^{ikx}\left[\frac{e^{ibt} - e^{-k^2 t}}{k^2 + ib} - \frac{e^{-ibt} - e^{-k^2 t}}{k^2 - ib}\right]\Bigg\} dk. \end{aligned} \tag{3.10}$$

In order to have rapid convergence for large k, we choose a path which for large k aligns with the directions of the steepest descent which for the current problem are $\arg k = \pm\frac{\pi}{8}$. For convenience we choose for $\mathcal{L}$ to be a hyperbola. Furthermore, we use a simple transformation to map this hyperbola to the real axis.

In general suppose that $\mathcal{L}$ is a hyperbola which asymptotes to $\arg k \to \alpha$ and $\arg k \to \pi - \alpha$ as $|k| \to \infty$. Then the following transformation maps $\mathcal{L}$ to the real line:

$$k(\theta) = i \sin(\alpha - i\theta). \tag{3.11}$$

Indeed,

$$k(\theta) = \frac{1}{2}[e^{i\alpha}e^{\theta} - e^{-i\alpha}e^{-\theta}],$$

thus $\arg k \to \alpha$ as $\theta \to \infty$ and $\arg k \to \pi - \alpha$ as $\theta \to -\infty$, and the real θ-axis is mapped to $\mathcal{L}$.

Using the transformation (3.11), with $\alpha = \frac{\pi}{8}$, (3.10) becomes

$$\begin{aligned} q(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} & e^{ik(\theta)x - k^2(\theta)t}\left[\frac{1}{(ik(\theta) + a^2)^2} - \frac{1}{(-ik(\theta) + a^2)^2}\right] \\ & -k(\theta)e^{ik(\theta)x}\left[\frac{e^{ibt} - e^{-k(\theta)^2 t}}{k(\theta)^2 + ib} - \frac{e^{-ibt} - e^{-k(\theta)^2 t}}{k(\theta)^2 - ib}\right]\cos(\alpha - i\theta)d\theta. \end{aligned} \tag{3.12}$$

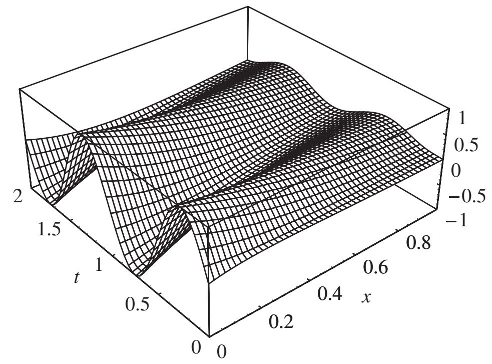


Figure 3.1. *The solution of the heat equation displayed on* $x \in [0, 1]$ *and* $t \in [0, 2]$.

Equation (3.12) defines an ordinary integral with an exponentially decaying integrand as $\theta \to \pm\infty$. Any language with a built-in numerical integrator, such as *Mathematica, Maple*, or MATLAB, can provide a simple approach to evaluating the integral. For example, using *Mathematica* and its NIntegrate command, only four lines of code is needed to evaluate and plot (3.12) for any x and t; see Figure 3.1.

3.2 The Equation $q_t + q_{xxx} = 0$ on the Half-Line

Consider the Dirichlet problem of (1.22a) on the half-line. Then using (1.16) with $\tilde{g}$ defined by (1.47) we find

$$\begin{aligned} q(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx+ik^3t}\hat{q}_0(k)dk \\ &-\frac{1}{2\pi}\int_{\partial D^+} e^{ikx+ik^3t}\left[3k^2 G_0(-ik^3,t) - \alpha\hat{q}_0(\alpha k) - \alpha^2\hat{q}_0(\alpha^2 k)\right]dk, \end{aligned} \tag{3.13}$$

$$\alpha = e^{\frac{2i\pi}{3}}, \quad 0 < x < \infty, \quad t > 0,$$

where the contour ∂D^+ is depicted in Figure 1.3. The term involving $\tilde{G}_0$ can be treated as the corresponding term of the heat equation. However, since the real axis is *not* surrounded by the domain satisfying $\mathrm{Re}(-ik^3) > 0$, it is *not* possible, by splitting $\hat{q}_0(k)$, to deform the real axis to a contour in the lower half of the complex k-plane. On the other hand, if $q_0(x)$ belongs to a restricted class, then the real axis can be deformed to the same contour that ∂D^+ will be deformed to in the upper half plane; similar considerations apply to the terms involving $\hat{q}_0(\alpha k)$ and $\hat{q}_0(\alpha^2 k)$.

Example 3.2 Let $q_0(x)$ and $g_0(t)$ be defined by (3.8). Then (3.13) becomes

$$\begin{aligned} q(x,t) = \frac{1}{2\pi}\int_{\mathcal{L}} \Bigg\{ & e^{ikx+ik^3t}\left[\frac{1}{(ik+a^2)^2} + \frac{\alpha}{(i\alpha k+a^2)^2} + \frac{\alpha^2}{(i\alpha^2 k+a^2)^2}\right] \\ & + \frac{3k^2}{2}e^{ikx}\left[\frac{e^{ibt}-e^{ik^3t}}{b-k^3} - \frac{e^{-ibt}-e^{ik^3t}}{b+k^3}\right]\Bigg\} dk, \qquad \alpha = e^{\frac{2i\pi}{3}}. \end{aligned} \tag{3.14}$$

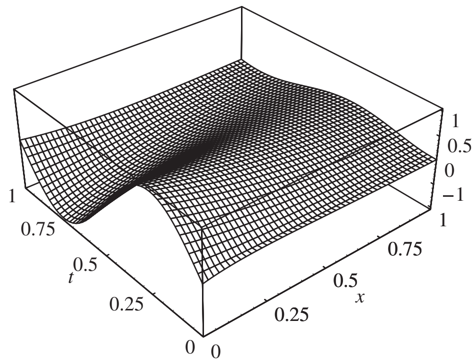


Figure 3.2. *The solution of* (1.22a) *displayed on* $x \in [0, 1]$ *and* $t \in [0, 1]$.

After using the substitution $k(\theta) = i \sin(\pi/6 - i\theta)$, the built-in numerical integrator in *Mathematica* can be used to evaluate the RHS of (3.14); see Figure 3.2.

3.3 The Equation $q_t - q_{xxx} = 0$ on the Half-Line

The solution of the canonical problem of (1.23a) on the half-line is given by (1.16) with $\tilde{g}$ defined by (1.51), where the contours ∂D_1^+ and ∂D_2^+ are depicted in Figure 1.4. The rays $\arg k = \pi/3$ and $2\pi/3$ can be deformed to the unshaded domain of the upper half complex k-plane of Figure 1.4, but if $\exp[ikx]$ and $\exp[-ik^3 t]$ are treated *separately*, the real axis *cannot* be deformed to the unshaded domain of the lower half of the complex k-plane, since $\exp[ikx]$ is unbounded for $\operatorname{Im} k < 0$. However, this problem can be bypassed if the term $\exp[ikx - ik^3 t]$ is treated as a *single* term as is done in the following example.

Example 3.3 Let

$$q_0(x) = x^2 e^{-a^2 x}, \quad g_0(t) = h_0(t) = 0, \tag{3.15}$$

where a is a positive real number. Then

$$\hat{q}_0(k) = \frac{2}{(ik + a^2)^3}. \tag{3.16}$$

Hence,

$$q(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{ikx - ik^3 t}}{(ik + a^2)^3} dk - \frac{1}{\pi}\int_{\partial D_1^+} \frac{e^{ikx - ik^3 t}}{(i\alpha k + a^2)^3} dk - \frac{1}{\pi}\int_{\partial D_2^+} \frac{e^{ikx - ik^3 t}}{(i\alpha^2 k + a^2)^3} dk. \tag{3.17}$$

The exponential $\exp[ikx - ik^3 t]$ is bounded, provided that $\operatorname{Re}[ikx - ik^3 t] \le 0$, i.e.,

$$-tk_I \left[\frac{x}{t} - 3k_R^2 + k_I^2\right] \le 0.$$

This domain is depicted in the shaded region of Figure 3.3, where the solid curves are defined by $x/t - 3k_R^2 + k_I^2 = 0$.

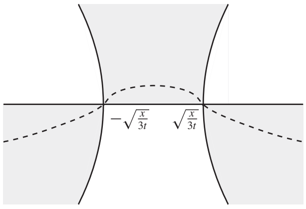


Figure 3.3. *The contours of integration of* (1.23a).

Thus, the real axis, as well as the contours ∂D_1^+ and ∂D_2^+ can be deformed to contours in the shaded region. The broken curve in Figure 3.3 depicts such a deformation of the real axis. These deformed contours can be mapped to the real θ-axis by using the following change of variables.

For the real axis,

$$k = -i \sin\left(\frac{\pi}{6} + i\theta\right) + \frac{i}{6}\sqrt{9 + 4\frac{x}{t}}. \tag{3.18}$$

For ∂D_1^+,

$$k = e^{\frac{i\pi}{6}} \sin\left(\frac{\pi}{6} - i\theta\right) + \sqrt{\frac{x}{3t}} - \frac{e^{\frac{i\pi}{6}}}{2}. \tag{3.19}$$

For ∂D_2^+,

$$k = -e^{-\frac{i\pi}{6}} \sin\left(\frac{\pi}{6} - i\theta\right) - \sqrt{\frac{x}{3t}} + \frac{e^{-\frac{i\pi}{6}}}{2}. \tag{3.20}$$

Indeed, for the deformation of the real axis, we let

$$k = -i \sin\left(\frac{\pi}{6} + i\theta\right) + ic, \quad -\infty < \theta < \infty, \tag{3.21}$$

where c is to be determined. As $\theta \to -\infty$, $k \to -|k| \exp[i\pi/6]$ and as $\theta \to +\infty$, $k \to |k| \exp[-i\pi/6]$, and thus (3.21) has the correct asymptotic behavior. We must choose c such that for a specific real value of θ which will be denoted by φ, the deformed curve goes through $\pm\sqrt{x/3t}$. This implies

$$\pm\sqrt{\frac{x}{3t}} = \frac{\sqrt{3}}{4}(e^{\varphi} - e^{-\varphi}), \quad c = \frac{1}{4}(e^{\varphi} + e^{-\varphi}).$$

Taking the square of these equations we find

$$\frac{1}{9}\left(\frac{x}{t}\right) = \frac{e^{2\varphi} + e^{-2\varphi} - 2}{16}, \quad c^2 = \frac{1}{16}(e^{2\varphi} + e^{-2\varphi} + 2).$$

Adding these equations we find

$$c^2 = \frac{1}{4} + \frac{1}{9}\frac{x}{t}. \tag{3.22}$$

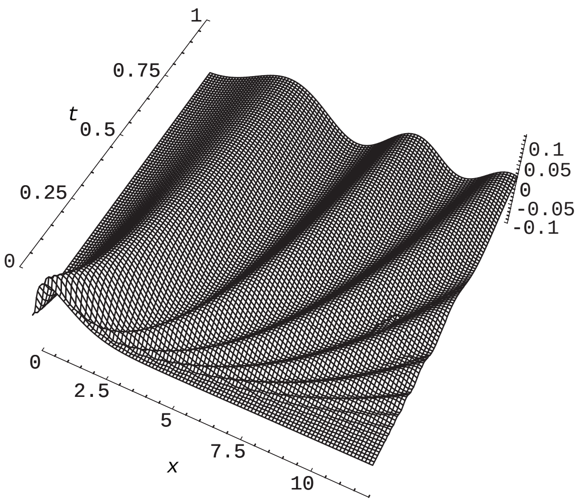


Figure 3.4. *The solution of* (1.23a) *displayed on* $x \in [0, 12]$ *and* $t \in [0, 1]$.

For the deformation of ∂D_1^+, we let

$$k = e^{\frac{i\pi}{6}} \sin\left(\frac{\pi}{6} - i\theta\right) + c,$$

where c is to be determined. As $\theta \to -\infty$, $k \to |k|i$, and as $\theta \to +\infty$, $k \to |k|\exp(-i\pi/6)$, and thus (3.19) has the correct asymptotic behavior. We must choose c such that when $\theta = 0$, $k = \sqrt{x/3t}$. This implies

$$\sqrt{\frac{x}{3t}} = \frac{e^{\frac{i\pi}{6}}}{2} + c,$$

which fixes c.

For the deformation of ∂D_2^+, proceeding as with ∂D_1^+ we find (3.20).

In this case there is an additional numerical difficulty due to the pole at $k = ia^2$: If we fix x and let t become small, the contour is shifted upwards, interacting with the pole and giving rise to a spurious contribution.

This difficulty can be overcome by subtracting the pole to obtain a singularity-free integrand. This is done by subtracting from the function, $\frac{e^{ikx-ik^3t}}{(ik+a^2)^3}$, a function that decays in the lower half-plane, and which has the same pole character. This function can be derived by requiring that the coefficients (a_{-1}, a_{-2}, a_{-3}) for the three components of the pole in the Laurent expansion of the integrand vanish; such a function is given in [3]. After this difficulty is bypassed then the solution can be evaluated numerically using *Mathematica*; see Figure 3.4.

Remark 3.1. One of the striking features of the solution depicted in Figure 3.4 is the wave pattern that emanates from the corner of the domain, $x = 0$, $t = 0$. This phenomenon results from the fact that unless the initial and boundary data are compatible for all orders (i.e., unless they satisfy an infinite set of compatibility conditions), the solution will feature irregularities that emanate from the corner and behave according to the nature of the given PDE (i.e., diffuse for the heat equation and propagate for the current problem) [80]. It is assumed in our analysis that the initial condition (IC) and the boundary condition (BC) match at the corner. The next order compatibility is $q(BC)_t = q(IC)_{3x}$, which is violated in this example since $q(BC)_t = 0$ and $q(IC)_{3x} = -27/2$. All higher order conditions are derived by taking derivatives with respect to the time of the PDE and substituting in the IC and the BC. An extensive study on the nature of such singularities and their numerical implications for dissipative, dispersive, and convective PDEs is given in [81], [82], [83]. The new method can handle these singularities in an efficient way.

Remark 3.2. For evolution PDEs, the main advantage of the new numerical method is that it can be used to compute solutions at arbitrary points in the (x, t)-plane. Neither time stepping nor spatial discretization is required. In this respect, the new method has some similarities with the Laplace transform technique [84]. However, in addition to the difficulties with the Laplace transform discussed in the introduction, we also note that the explicit and analytic dependence on k of the novel formulae used here allows us to deform contours which in turn yields exponentially decaying integrals. This yields efficient numerical computations, which should be contrasted with the numerical computation of the inverse Laplace transform [84].

Remark 3.3. It appears that the semi-analytical nature of the new method also has a pedagogical advantage: The usual numerical techniques of finite differences, finite element, and spatial discretizations [85] are constructed independently of the analytical treatment of the given PDE. This often raises questions about the reason for teaching the students analytical techniques. This should be contrasted with the new method, where the numerical integration is the last step of an approach which is based on the analytical treatment of the given PDE.

Part II

Analytical Inversion of Integrals

Chapter 4

From PDEs to Classical Transforms

The implementation of the new transform method to evolution PDEs was presented in Part I. The implementation of this method to several elliptic PDEs will be discussed in Part IV. The new method is *not* based on the derivation or even the existence of classical transforms and actually is applicable even to cases where classical transforms do *not* exist.

If a given boundary value problem *can* be solved by a classical transform, the new method provides an alternative approach to deriving this transform. Indeed, recall that the new transform method yields the solution q as an integral in the complex k-plane. If there exists a classical transform representation, then it is possible, using contour deformation and Cauchy's theorem, to rewrite q in terms of a series involving the relevant residues plus an integral along the real axis. This representation defines the inverse of the associated classical transform. The advantage of this approach is that it bypasses the difficult problem of completeness. The general approach will be illustrated with the aid of the following examples.

Example 4.1 (a generalization of the cosine and sine transformations). Define the following generalization of the sine and cosine transforms:

$$\hat{f}(k) = \int_0^\infty \left(e^{-ikx} + \frac{k - i\gamma}{k + i\gamma} e^{ikx} \right) f(x) dx, \quad k \in \mathbb{R}, \tag{4.1}$$

where γ is a finite real constant and the smooth function $f(x)$ has sufficient decay as $x \to \infty$. Then

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^\infty e^{ikx} \hat{f}(k) dk - 2\gamma e^{\gamma x} H(-\gamma) \int_0^\infty e^{\gamma\xi} f(\xi) d\xi, \quad x > 0, \tag{4.2}$$

where $H(\cdot)$ denotes the Heaviside function.

Indeed, the solution of the Robin problem of the heat equation on the half-line is given by (1.38b). For $\gamma > 0$, by deforming the contour ∂D^+ to the real axis we find

$$\begin{aligned} q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^2t}\left[\int_0^{\infty}\left(e^{-ik\xi}+\frac{k-i\gamma}{k+i\gamma}e^{ik\xi}\right)q_0(\xi)d\xi\right]dk \\ -\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^2t}\left[\frac{2k}{k+i\gamma}\int_0^t e^{k^2s}g_R(s)ds\right]dk. \end{aligned} \tag{4.3}$$

Evaluating this equation at $t = 0$ and renaming $q_0(x)$ by $f(x)$ we find (4.2) for $\gamma > 0$. If $\gamma < 0$, then we also have the second term in the RHS of (4.2).

Equations (4.1) and (4.2) contain the cosine and sine transforms as particular cases. Indeed, if $\gamma = 0$, (4.1) and (4.2) become

$$\hat{f}(k) = 2\int_0^{\infty}\cos(kx)f(x)dx, \quad k \in \mathbb{R}, \tag{4.4}$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\hat{f}(k)dk, \quad x > 0. \tag{4.5}$$

Using the fact that $\hat{f}(k)$ is an even function, (4.4) and (4.5) can be rewritten in the form of the classical cosine transform ($\hat{f}/2 \to \hat{f}$),

$$\hat{f}(k) = \int_0^{\infty}\cos(kx)f(x)dx, \quad k > 0, \tag{4.6}$$

$$f(x) = \frac{2}{\pi}\int_0^{\infty}\cos(kx)\hat{f}(k)dk, \quad x > 0. \tag{4.7}$$

Similarly, letting $\gamma \to \infty$, (4.1) and (4.2) become

$$\hat{f}(k) = -2i\int_0^{\infty}\sin(kx)f(x)dx, \quad k > 0, \tag{4.8}$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\hat{f}(k)dk, \quad x > 0. \tag{4.9}$$

Using the fact that $\hat{f}(k)$ is an odd function, (4.8) and (4.9) can be rewritten in the form of the classical sine transform ($\frac{\hat{f}}{-2i} \to \hat{f}$),

$$\begin{aligned} \hat{f}(k) &= \int_0^{\infty}\sin(kx)f(x)dx, \quad k > 0, \\ \hat{f}(x) &= \frac{2}{\pi}\int_0^{\infty}\sin(kx)\hat{f}(k)dk, \quad x > 0. \end{aligned} \tag{4.10}$$

Example 4.2 (a generalization of the cosine and sine series). Define the following generalization of the sine and cosine transform:

$$\hat{f}_n = \int_0^L \left(e^{ik_n x} + \frac{k_n + i\gamma_1}{k_n - i\gamma_1} e^{-ik_n x} \right) f(x) dx, \quad L > 0, \tag{4.11a}$$

where $f(x)$ is a smooth function and k_n satisfies

$$\frac{\tan(k_n L)}{k_n} = \frac{\gamma_1 + \gamma_2}{k_n^2 - \gamma_1 \gamma_2} \text{ and } k_0 = 0, \gamma_1 \geq 0, \gamma_2 \geq 0. \tag{4.11b}$$

Then, for $0 < x < L$,

$$f(x) = -2i \sum_{n=1}^{\infty} \frac{1}{\Delta'(k_n)} (ik_n - \gamma_2) e^{-ik_n L} [ik_n \cos(k_n x) + i\gamma_1 \sin(k_n x)] \hat{f}_n, \tag{4.12}$$

for $\gamma_1 > 0, \gamma_2 > 0$, where $\Delta(k)$ is defined by (2.14c). If $\gamma_1 = \gamma_2 = 0$, then the RHS of (4.12) contains the term $\frac{1}{2L}\hat{f}_0(0)$.

Indeed, the solution of the Robin problem for the heat equation on the finite interval is given by (2.28). Evaluating this equation at $t = 0$, renaming $q_0(x)$ as $f(x)$, taking as common factor the term

$$-\frac{2i}{\Delta'(k_n)} (ik_n - \gamma_2) e^{-ik_n L} [ik_n \cos(k_n x) + i\gamma_1 \sin(k_n x)],$$

and using that (see (2.26a))

$$\frac{k_n - i\gamma_2}{k_n + i\gamma_2} e^{2ik_n L} = \frac{k_n + i\gamma_1}{k_n - i\gamma_1},$$

(2.28) yield (4.12).

Equations (4.11) and (4.12) contain the cosine and sine series as particular cases. Indeed, if $\gamma_1 = \gamma_2 = 0$, (4.11) yield

$$\hat{f}_n = 2 \int_0^L \cos(k_n x) f(x) dx, \quad k_n = \frac{n\pi}{L}. \tag{4.13}$$

Furthermore, (4.12) yields

$$f(x) = \frac{1}{2L} \hat{f}_0 + \frac{1}{L} \sum_{n=1}^{\infty} \cos(k_n x) \hat{f}_n. \tag{4.14}$$

Similarly, letting $\gamma_1 \to \infty$, $\gamma_2 \to \infty$, equations (4.11) yield

$$\hat{f}_n = 2i \int_0^L \sin(k_n x) f(x) dx, \quad k_n = \frac{n\pi}{L}, \tag{4.15}$$

and (4.12) yields

$$f(x) = -\frac{i}{L} \sum_{n=1}^{\infty} \sin(k_n x) \hat{f}_n. \tag{4.16}$$

Remark 4.1. A variety of boundary value problems for the biharmonic equation in a semi-infinite strip are solved in [33]. A subclass of these problems can be solved by the classical Papkovich–Fadle eigenfunction *series* expansion. The relevant solutions can be rederived from the integral representations derived in [33] by employing Cauchy's theorem. In this way, it is possible to derive the general representation result involving the Papkovich–Fadle eigenfunctions, avoiding the difficult problem of completeness.

Chapter 5

Riemann–Hilbert and *d*-Bar Problems

Consider the integral

$$\Phi(z) = \frac{1}{2\pi i}\int_{\mathcal{L}} \frac{\varphi(\tau)}{\tau - z}d\tau, \tag{5.1}$$

where $\mathcal{L}$ is a smooth finite curve ($\mathcal{L}$ may be an arc or a closed contour) and $\varphi(\tau)$ is a function satisfying the *Hölder condition* on $\mathcal{L}$; that is, for any two points τ and τ_1 on $\mathcal{L}$,

$$|\varphi(\tau) - \varphi(\tau_1)| \leq \wedge|\tau - \tau_1|^{\lambda}, \quad \wedge > 0, \quad 0 < \lambda \leq 1. \tag{5.2}$$

If $\lambda = 1$, the Hölder condition becomes the so-called Lipschitz condition. For example, a differentiable function $\varphi(\tau)$ satisfies the Hölder (Lipschitz) condition with $\lambda = 1$. The integral (5.1) is well defined and $\Phi(z)$ is analytic, provided that z is not on $\mathcal{L}$. We also note that if z is not on $\mathcal{L}$, then

$$\Phi(z) = \left(-\frac{1}{2i\pi}\int_{\mathcal{L}} \varphi(\tau)d\tau\right)\frac{1}{z} + O\left(\frac{1}{z^2}\right), \quad |z| \to \infty, \quad z \notin \mathcal{L}.$$

However, if z is on $\mathcal{L}$, this integral becomes ambiguous; to give it a unique meaning we must know how z approaches $\mathcal{L}$. We denote by $+$ the region that is on the left of the positive direction of $\mathcal{L}$ and by $-$ the region on the right; see Figure 5.1. It turns out that $\Phi(z)$ has a limit $\Phi^+(t)$, t on $\mathcal{L}$, when z approaches $\mathcal{L}$ along a curve entirely in the $+$ region. Similarly, $\Phi(z)$ has a limit $\Phi^-(t)$, when z approaches $\mathcal{L}$ along a curve entirely in the $-$ region. These limits, which play a fundamental role in the theory of Riemann–Hilbert (RH) problems, are given by the so-called Plemelj formulae.

5.1 Plemelj Formula

Let $\mathcal{L}$ be a smooth finite contour (closed or open) and let $\varphi(\tau)$ satisfy a Hölder condition on $\mathcal{L}$. Then the Cauchy-type integral $\Phi(z)$, defined in (5.1), has the limiting values $\Phi^+(t)$ and $\Phi^-(t)$ as z approaches $\mathcal{L}$ from the left and the right, respectively, and t is not an endpoint of $\mathcal{L}$. These limits are given by

$$\Phi^{\pm}(t) = \pm\frac{1}{2}\varphi(t) + \frac{1}{2\pi i}⨍_{\mathcal{L}} \frac{\varphi(\tau)}{\tau - t}d\tau. \tag{5.3$^{\pm}$}$$

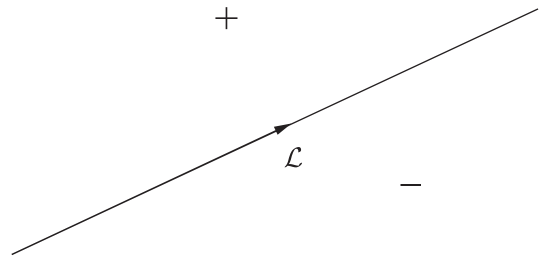


Figure 5.1. *Regions "+" and "−" on either side of* $\mathcal{L}$.

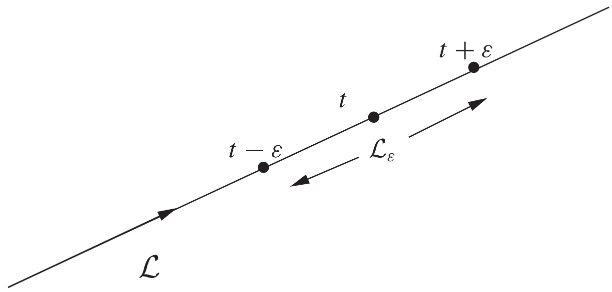


Figure 5.2. *The curve* $\mathcal{L}_\varepsilon$.

In these equations, $\fint$ denotes the principal value integral defined by

$$\fint_{\mathcal{L}} \frac{\varphi(\tau)d\tau}{\tau - t} = \lim_{\varepsilon\to 0} \int_{\mathcal{L}-\mathcal{L}_\varepsilon} \frac{\varphi(\tau)d\tau}{\tau - t}, \tag{5.4}$$

where $\mathcal{L}_\varepsilon$ is the part of $\mathcal{L}$ that has length 2ε and is centered around t, as depicted in Figure 5.2.

The derivation of the Plemelj formula is straightforward if $\varphi(\tau)$ is analytic in the neighborhood of $\mathcal{L}$; see [17]. The derivation in the case that $\varphi(\tau)$ is Hölder is rather complicated; see [86].

In the above formulation we have assumed that $\mathcal{L}$ is a finite contour; otherwise $\varphi(\tau)$ must satisfy an additional condition. Suppose, for example, that $\mathcal{L}$ is the real axis; then we assume that $\varphi(\tau)$ satisfies a Hölder condition for all finite τ, and that as $t \to \pm\infty$, we have $\varphi(\tau) \to \varphi(\infty)$, where

$$|\varphi(\tau) - \varphi(\infty)| < \frac{M}{|\tau|^\mu}, \quad M > 0, \quad \mu > 0. \tag{5.5}$$

Equations (5.3) are equivalent to

$$\Phi^+(t) - \Phi^-(t) = \varphi(t), \quad \Phi^+(t) + \Phi^-(t) = \frac{1}{\pi i}\fint \frac{\varphi(\tau)}{\tau - t} d\tau. \tag{5.6}$$

Equations (5.6) are the main formulae needed for the solution of a scalar RH problem. In this respect we first introduce the following definitions.

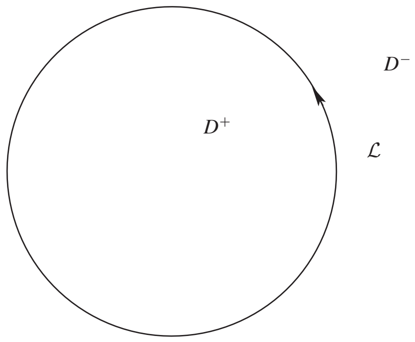


Figure 5.3. *Simple closed contour $\mathcal{L}$ and the "+" and "−" regions.*

(1) Let $\mathcal{L}$ be a simple, smooth, closed contour dividing the complex z-plane into two regions D^+ and D^-, where the positive direction of $\mathcal{L}$ will be taken as that for which D^+ is on the left; see Figure 5.3. A scalar function $\Phi(z)$ defined in the entire plane, except for points on $\mathcal{L}$, will be called *sectionally analytic* if (a) the function $\Phi(z)$ is analytic in each of the regions D^+ and D^- except, perhaps, at $z = \infty$, and (b) the function $\Phi(z)$ is sectionally continuous with respect to $\mathcal{L}$; i.e., as z approaches any point t on $\mathcal{L}$ along any path which lies wholly in either D^+ or D^-, the function $\Phi(z)$ approaches a definite limiting value $\Phi^+(t)$ or $\Phi^-(t)$, respectively.

It then follows from a result due to Painlevé that $\Phi(z)$ is continuous in the closed region $D^+ + \mathcal{L}$ if it is assigned the value $\Phi^+(t)$ on $\mathcal{L}$. A similar statement applies for the region $D^- + \mathcal{L}$.

(2) The sectionally analytic function $\Phi(z)$ is said to have degree m at infinity, where m is a positive integer, if

$$\Phi(z) \sim c_m z^m + O(z^{m-1}) \text{ as } z \to \infty, \quad c_m \text{ a nonzero constant}, \quad z \notin \mathcal{L}. \tag{5.7}$$

The scalar *homogeneous* RH problem for closed contours is formulated as follows: Given a closed contour $\mathcal{L}$ and a function $g(t)$ which is Hölder on $\mathcal{L}$ with $g(t) \neq 0$ on $\mathcal{L}$, find a sectionally analytic function $\Phi(z)$, with finite degree m at infinity, such that

$$\Phi^+(t) = g(t)\Phi^-(t) \text{ on } \mathcal{L}, \tag{5.8}$$

where $\Phi^\pm(t)$ are the boundary values of $\Phi(z)$ on $\mathcal{L}$.

The scalar *inhomogeneous* RH problem is

$$\Phi^+(t) = g(t)\Phi^-(t) + f(t), \quad t \text{ on } \mathcal{L}, \tag{5.9}$$

where $f(t)$ is also Hölder on $\mathcal{L}$.

The solutions of the RH problems (5.8) and (5.9) is presented in [17]. In this book we will need only the solution of the following simple RH problem (corresponding to (5.9) with $g = 1$):

$$\Phi^+(t) - \Phi^-(t) = f(t), \quad t \text{ on } \mathcal{L},$$
$$\Phi(z) = O\left(\frac{1}{z}\right), \quad z \to \infty, \quad z \notin \mathcal{L}. \tag{5.10}$$

The first of equations (5.6), together with Liouville's theorem, immediately implies that the unique solution of this RH problem is given by

$$\Phi(z) = \frac{1}{2i\pi} \int_{\mathcal{L}} \frac{f(\tau)}{\tau - z} d\tau. \tag{5.11}$$

In many applications $\mathcal{L}$ is the real axis, and then the Plemelj formulae (5.3) become

$$\Phi^{\pm}(x) = \pm\frac{1}{2}\varphi(x) + \frac{1}{2i}(H\varphi)(x), \tag{5.12}$$

where H denotes the Hilbert transform

$$(Hf)(x) = \frac{1}{\pi} \fint_{-\infty}^{\infty} \frac{f(\xi)}{\xi - x} dx. \tag{5.13}$$

We recall (see [87]) that the map $f \to Hf$ is bounded in L^p for all $1 < p < \infty$ (this map is *not* bounded in L^1). Actually, a convenient space for the study of an RH problem is H^1: It can be shown that if $f \in H^1(\mathcal{L})$, then (see [16])

$$\sup_{z \in \mathbb{C}\setminus\mathcal{L}} \left| \frac{1}{2i\pi} \int_{\mathcal{L}} \frac{f(\tau)}{\tau - z} d\tau \right| \le \|f\|^2_{H^1(\mathcal{L})}, \tag{5.14}$$

where

$$\|f\|^2_{H^1(\mathcal{L})} = \int_{\mathcal{L}} \left(|f(\zeta)|^2 + |f'(\zeta)|^2 \right) d\zeta. \tag{5.15}$$

5.2 The d-Bar Problem

Equation (5.11) indicates that the sectionally analytic function $\Phi(z)$ defined by

$$\Phi(z) = \begin{cases} \Phi^+(z), & z \in D^+, \\ \Phi^-(z), & z \in D^-, \end{cases}$$

is analytic in the entire complex z-plane (including ∞), except for z on $\mathcal{L}$. The "departure from analyticity" is measured by the d-bar derivative $\partial/\partial\bar{z}$. Hence, $\partial\Phi/\partial\bar{z}$ has support only for $z \in \mathcal{L}$. A natural generalization of the RH problem defined by (5.10) is the problem of determining a function whose d-bar derivative has support in a two-dimensional domain:

$$\frac{\partial\Phi}{\partial\bar{z}}(z, \bar{z}) = f(z, \bar{z}), \quad z \in D \subset \mathbb{R}^2. \tag{5.16}$$

The problem of determining Φ in terms of f and of the value of Φ on the boundary of D (denoted by ∂D) is called a d-bar problem.

In the same way that the Plemelj formulae play a crucial role in solving an RH problem, the following formula provides the basis of the solution of a d-bar problem:

$$\Phi(z, \bar{z}) = \frac{1}{2i\pi} \int_{\partial D} \Phi(\zeta, \bar{\zeta}) \frac{d\zeta}{\zeta - z} + \frac{1}{2i\pi} \int\int_{D} \frac{\partial\Phi}{\partial\bar{\zeta}}(\zeta, \bar{\zeta}) \frac{d\zeta \wedge d\bar{\zeta}}{\zeta - z}. \tag{5.17}$$

This equation, which is known as the d-bar, the Cauchy–Green, or the Pompeiu formula, is a direct consequence of the following identity for the smooth function F:

$$\int_{\partial D} F(z, \bar{z})dz = \int\int_D \frac{\partial F}{\partial \bar{z}} d\bar{z} \wedge dz. \tag{5.18}$$

The derivation of (5.17) starting with (5.18) can be found in [17]. Equation (5.18) follows immediately from the Poincaré lemma,

$$\int_{\partial D} W = \int\int_D dW, \tag{5.19}$$

with $W = Fdz$, which implies

$$dW = \frac{\partial F}{\partial \bar{z}} d\bar{z} \wedge dz.$$

Alternatively, (5.18) follows from the usual Green's theorem,

$$\int_{\partial D} (udx + vdy) = \int\int_D (v_x - u_y)dxdy,$$

with $F = u + iv$ and $z = x + iy$, which imply

$$dz = dx + idy, \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\partial_x + i\partial_y).$$

Equations (5.16) and (5.17) imply that the solution of the d-bar problem (5.16) is given by (5.17) with $\Phi_{\bar{\zeta}}$ replaced by f.

For a finite domain D, one requires only that f has sufficient smoothness. However, in many applications D is the entire complex plane and also $\Phi \sim 1$ as $z \to \infty$; i.e., Φ satisfies

$$\frac{\partial \Phi}{\partial \bar{z}} = f(z, \bar{z}), \quad z \in \mathbb{C},$$

$$\Phi = 1 + O\left(\frac{1}{z}\right), \quad z \to \infty. \tag{5.20}$$

Then, replacing Φ by $\Phi - 1$ in (5.17) and noting that the integral over ∂D of $\Phi - 1$ vanishes, we find that the unique solution of (5.20) is given by

$$\Phi = 1 + \int\int_{\mathbb{C}} \frac{f(\zeta, \bar{\zeta})}{\zeta - z} d\zeta \wedge d\bar{\zeta}, \tag{5.21}$$

where $\mathbb{C}$ denotes integration over the entire complex plane. The possible values of ζ for which this integral may have singularities are $\zeta = z$ and $\zeta = \infty$. It turns out that if $f \in L_{2+\varepsilon}$, where ε is arbitrarily small, the integral is well behaved as $\zeta \to z$, and if $f \in L_{2-\varepsilon}$, then the integral is well behaved as $\zeta \to \infty$. Thus a convenient class of functions is $f \in L_1 \cap L_\infty$.

In this case $\Phi \to 1$ as $z \to \infty$ and Φ is continuous. Also Φ satisfies $\frac{\partial \Phi}{\partial \bar{z}} = f$, but in a weak sense. In order for $\frac{\partial \Phi}{\partial \bar{z}}$ to exist in a strong sense one needs some smoothness conditions on f (for example, f_z and $f_{\bar{z}} \in L_1 \cap L_\infty$).

Remark 5.1. Let $\zeta = \xi + i\eta$. Then

$$d\zeta \wedge d\bar{\zeta} = (d\xi + id\eta) \wedge (d\xi - id\eta) = -2id\xi d\eta.$$

Thus, the term $d\zeta \wedge d\bar{\zeta}$ in (5.17) and (5.21) can be replaced by $-2id\xi d\eta$.

Chapter 6

The Fourier Transform and Its Variations

In this chapter, starting from a given eigenvalue equation we introduce the main ideas and techniques needed for the construction of the associated transform pair $\{f, \hat{f}\}$. The relevant analysis, which will be referred to as the *spectral analysis*, involves two main steps:

(i) Solve the given eigenvalue equation in terms of f. If k denotes the eigenvalue parameter, this involves constructing a solution μ (of the given eigenvalue equation) which is *bounded* for *all* complex values of k. This problem will be referred to as the *direct problem*.

(ii) Using the fact that μ is bounded for all complex k, construct an alternative representation of μ which (instead of depending on f) depends on some "spectral function" of f denoted by $\hat{f}$. This problem will be referred to as the *inverse problem*.

It turns out that the inverse problem gives rise to either a Riemann–Hilbert (RH) or a $\bar{\partial}$ problem. Indeed, for certain eigenvalue problems the function μ is *sectionally analytic* in k; i.e., it has different representations in different domains of the complex k-plane and each of these representations is analytic. In this case, if the "jumps" of these representations across the different domains can be expressed in terms of $\hat{f}$, then it is possible to reconstruct μ as the solution of an RH problem which is uniquely defined in terms of $\hat{f}$. However, for a large class of eigenvalue problems, there exists a domain in the complex k-plane where μ is *not* analytic. In this case, if $\partial\mu/\partial\bar{k}$ can be expressed in terms of $\hat{f}$, then μ can be reconstructed through the solution of a $\bar{\partial}$ problem which is uniquely defined in terms of $\hat{f}$.

As it was mentioned in the introduction, the derivation of classical transform pairs through the integration in the complex λ-plane of an appropriate Green's function is based on the assumption that the Green's function is an *analytic* function of λ. This corresponds to the case that μ is sectionally analytic. Therefore, the approach presented in this chapter has the advantages that it not only provides a simpler approach to deriving classical transforms (avoiding the problem of completeness), but it also can be applied to problems where the associated Green's function is *not* an analytic function of λ.

Example 6.1 (the Fourier transform). The classical Fourier transform can be derived through the spectral analysis of the following eigenvalue equation for the function $\mu(x,k)$:

$$\mu_x - ik\mu = f(x), \quad x \in \mathbb{R}, \quad k \in \mathbb{C}, \tag{6.1}$$

where $f(x)$ is an arbitrary function with appropriate smoothness and decay.

The solution of the direct problem involves solving (6.1) for μ in terms of f for all $k \in \mathbb{C}$. The solution of this problem is elementary, since (6.1) is a first order ODE,

$$\frac{d}{dx}\left(\mu e^{-ikx}\right) = e^{-ikx} f(x). \tag{6.2}$$

In order to solve this equation, we can integrate with respect to x either from $-\infty$ or from $+\infty$. Actually, we will need *both* these solutions, which we will denote by μ^+ and μ^-, respectively,

$$\mu^+(x,k) = \int_{-\infty}^{x} e^{ik(x-\xi)} f(\xi)d\xi, \quad x \in \mathbb{R}, \quad \operatorname{Im} k \geq 0, \tag{6.3a}$$

$$\mu^-(x,k) = -\int_{x}^{\infty} e^{ik(x-\xi)} f(\xi)d\xi, \quad x \in \mathbb{R}, \quad \operatorname{Im} k \leq 0. \tag{6.3b}$$

The real part of $ik(x-\xi)$ equals $-k_I(x-\xi)$, and since $x-\xi \geq 0$ for μ^+, it follows that μ^+ is bounded for $\operatorname{Im} k \geq 0$. Similarly, μ^- is bounded for $\operatorname{Im} k \leq 0$. Furthermore, both μ^+ and μ^- depend *analytically* on k. Therefore the following sectionally analytic function of k solves (6.1) for all $x \in \mathbb{R}$:

$$\mu = \begin{cases} \mu^+, & \operatorname{Im} k \geq 0, \\ \mu^-, & \operatorname{Im} k \leq 0. \end{cases} \tag{6.4}$$

The solution of the inverse problem is also elementary due to the following facts:

(a) Equations (6.3) are both valid for $\operatorname{Im} k = 0$, i.e., for $k \in \mathbb{R}$. Hence, for k real by subtracting (6.3) we find the following "jump condition" for the sectionally analytic function μ, which is valid for all $x \in \mathbb{R}$:

$$\mu^+ - \mu^- = e^{ikx}\hat{f}(k), \quad k \in \mathbb{R}, \tag{6.5}$$

where $\hat{f}(k)$ is defined by

$$\hat{f}(k) = \int_{-\infty}^{\infty} e^{-ik\xi} f(\xi)d\xi, \quad k \in \mathbb{R}. \tag{6.6}$$

(b) Using integration by parts, (6.3) imply the following estimate for the behavior of μ at $k = \infty$:

$$\mu = O\left(\frac{1}{k}\right), \quad k \to \infty. \tag{6.7}$$

Equations (6.4), (6.5), (6.7) define a scalar RH problem in the variable k for the sectionally analytic function $\mu(x,k)$ (for this problem the variable x is a fixed parameter). The unique solution of this problem for all $x \in \mathbb{R}$ is

$$\mu(x,k) = \frac{1}{2i\pi}\int_{-\infty}^{\infty} \frac{e^{ilx}}{l-k}\hat{f}(l)dk, \quad k \in \mathbb{C}, \quad \operatorname{Im} k \neq 0. \tag{6.8}$$

Equations (6.3) and (6.4) express μ in terms of f (this is the solution of the direct problem), whereas (6.8) expresses μ in terms of $\hat{f}$ (this is the solution of the inverse problem). Using these two different representations for μ it is elementary to express f in terms of $\hat{f}$: Replacing μ in (6.1) by the RHS of (6.8) we find

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ilx} \hat{f}(l) dl, \quad x \in \mathbb{R}. \tag{6.9}$$

Equations (6.6) and (6.9) define the Fourier transform pair.

Example 6.2 (a variation of the Fourier transform). Let $\mu(t, k)$ satisfy

$$\mu_t - ik^2 \mu = k f(t), \quad 0 < t < T, \quad k \in \mathbb{C}, \tag{6.10}$$

where $f(t)$ is an arbitrary smooth function and T is a finite positive constant.

In analogy with (6.3) we now have

$$\mu^+(t, k) = k \int_0^t e^{ik^2(t-s)} f(s) ds, \;\; 0 < t < T, \;\; k \in D^+, \tag{6.11a}$$

$$\mu^-(t, k) = -k \int_t^T e^{ik^2(t-s)} f(s) ds, \;\; 0 < t < T, \;\; k \in D^-, \tag{6.11b}$$

where

$$D^+ = \left\{ k \in \mathbb{C}, \;\; \arg k \in \left[0, \frac{\pi}{2}\right] \cup \left[\pi, \frac{3\pi}{2}\right] \right\},$$

$$D^- = \left\{ k \in \mathbb{C}, \;\; \arg k \in \left[\frac{\pi}{2}, \pi\right] \cup \left[\frac{3\pi}{2}, 2\pi\right] \right\}. \tag{6.12}$$

Equations (6.11) imply that the sectionally analytic function μ, defined for all $0 < t < T$ by

$$\mu = \begin{cases} \mu^+, & k \in D^+, \\ \mu^-, & k \in D^-, \end{cases} \tag{6.13}$$

satisfies the following jump condition:

$$\mu^+ - \mu^- = e^{ik^2 t} k \hat{f}(k), \;\; 0 < t < T, \;\; k \in \{\mathbb{R} \cup i\mathbb{R}\}, \tag{6.14}$$

where $\hat{f}(k)$ is defined by

$$\hat{f}(k) = \int_0^T e^{-ik^2 s} f(s) ds, \quad k \in \{\mathbb{R} \cup i\mathbb{R}\}. \tag{6.15}$$

Equations (6.13) and (6.14), together with the estimate (6.7), imply that

$$\mu(t, k) = \frac{1}{2i\pi} \int_{\mathcal{L}} \frac{l e^{il^2 t} \hat{f}(l)}{l - k} dl, \;\; k \in \mathbb{C}, \;\; k \notin \{\mathbb{R} \cup i\mathbb{R}\}, \;\; 0 < t < T, \tag{6.16}$$

where $\mathcal{L}$ denotes the union of the real and imaginary axes with the orientation shown in Figure 6.1.

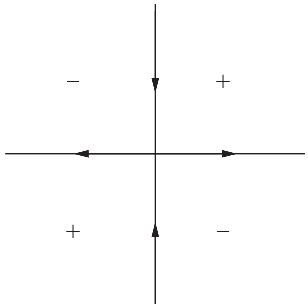


Figure 6.1. *The contour $\mathcal{L}$ for* (6.10).

Replacing μ in (6.10) by the RHS of (6.16) we find

$$f(t) = \frac{1}{2\pi}\int_{\mathcal{L}} e^{il^2 t} l \hat{f}(l)dl, \quad 0 < t < T, \tag{6.17}$$

as well as the equation

$$\int_{\mathcal{L}} e^{il^2 t} l^2 \hat{f}(l)dl = 0, \quad 0 < t < T. \tag{6.18}$$

The identity (6.18) is a direct consequence of the fact that $\hat{f}(l)$ is an even function.

Equation (6.17) can be rewritten in the form

$$f(t) = \frac{1}{\pi}\int_{\partial I} e^{il^2 t} l \hat{f}(l)dl, \quad 0 < t < T, \tag{6.19}$$

where ∂I denotes the boundary of the domain I with the orientation that I is to the left of the increasing direction, where

$$I = \{k \in \mathbb{C}, \arg k \in [0, \tfrac{\pi}{2}]\}.$$

The transform pair defined by (6.15) and (6.19) can be obtained by the usual Fourier transform pair using the change of variables $k^2 \to k$.

Example 6.3 (the Mellin transform). Let $\mu(\rho, k)$ satisfy

$$\rho\mu_\rho + k\mu = f(\rho), \quad \rho \in \mathbb{R}^+, \quad k \in \mathbb{C}, \tag{6.20}$$

where the smooth function $f(\rho)$ has appropriate decay as $\rho \to \infty$.

In analogy with (6.3) we now have

$$\mu^+ = \int_0^\rho \left(\frac{\tau}{\rho}\right)^k \frac{f(\tau)}{\tau} d\tau, \quad \rho \in \mathbb{R}^+, \quad \operatorname{Re} k \geq 0,$$

$$\mu^- = -\int_\rho^\infty \left(\frac{\tau}{\rho}\right)^k \frac{f(\tau)}{\tau} d\tau, \quad \rho \in \mathbb{R}^+, \quad \operatorname{Re} k \leq 0.$$

Hence,

$$\mu^+ - \mu^- = \frac{\hat{f}(k)}{\rho^k}, \quad \rho \in \mathbb{R}^+, \quad k \in i\mathbb{R},$$

where

$$\hat{f}(k) = \int_0^\infty \tau^{k-1} f(\tau) d\tau, \quad k \in i\mathbb{R}. \tag{6.21}$$

The solution of the inverse problem yields

$$\mu = -\frac{1}{2i\pi} \int_{-i\infty}^{i\infty} \frac{\rho^{-l} \hat{f}(l)}{l-k} dl, \quad \rho \in \mathbb{R}^+, \quad k \in \mathbb{C}, \quad \operatorname{Im} k \neq 0. \tag{6.22}$$

Replacing μ in (6.20) by the RHS of (6.22) we find

$$f(\rho) = \frac{1}{2i\pi} \int_{-i\infty}^{i\infty} \rho^{-l} \hat{f}(l) dl, \quad \rho \in \mathbb{R}^+. \tag{6.23}$$

Equations (6.21) and (6.23) define the Mellin transform.

Remark 6.1. Let $H_k^{(2)}(\rho)$ and $J_k(\rho)$ denote the second Hankel function and the Bessel function, respectively. It is straightforward to show that the spectral analysis of the ODE

$$\frac{d}{d\rho}\left(\frac{\mu(\rho,k)}{H_k^{(2)}(\rho)}\right) = k J_k(\rho) \frac{f(\rho)}{\rho}, \quad \rho \in \mathbb{R}^+, \quad k \in \mathbb{C},$$

yields the Kontorovich–Lebedev transform pair

$$\hat{f}(k) = \int_0^\infty \frac{J_k(\tau)}{\tau} f(\tau) d\tau, \quad k \in i\mathbb{R},$$

$$f(\rho) = -\frac{1}{2} \int_{-i\infty}^{i\infty} l H_l^{(2)}(\rho) \hat{f}(l) dl, \quad \rho \in \mathbb{R}^+.$$

Chapter 7

The Inversion of the Attenuated Radon Transform and Medical Imaging

The celebrated Radon transform provides the mathematical basis of computerized tomography (CT). Similarly, the attenuated Radon transform provides the mathematical basis of a new imaging technique of great significance, namely single photon emission computerized tomography (SPECT). Before discussing the mathematics of CT and SPECT we first present a brief introduction of these remarkable imaging techniques.

7.1 Computerized Tomography

In brain imaging, CT is the computer aided reconstruction of a mathematical function that represents the X-ray attenuation coefficient of the brain tissue (and is therefore related to its density). Let $f(x_1, x_2)$ denote the X-ray attenuation coefficient at the point (x_1, x_2). This means that X-rays transversing a small distance $\Delta\tau$ at (x_1, x_2) suffer a relative intensity loss $\Delta I/I = -f\Delta\tau$. Taking the limit and solving the resulting ODE we find $I_1/I_0 = \exp[-\int_L f d\tau]$, where L denotes the part of the line that transverses the tissue. Since I_1/I_0 is known from the measurements, the basic mathematical problem of CT is to reconstruct a function from the knowledge of its line integrals. The line integral of a function is called its Radon transform. In order to define this transform we introduce local coordinates: Let the line L make an angle θ with the positive x_1-axis. A point (x_1, x_2) on this line can be specified by the variables (ρ, τ), where ρ is the distance from the origin and τ is a parameter along the line; see Figure 7.1.

A unit vector $\underline{k}$ along L is given by $(\cos\theta, \sin\theta)$, and thus

$$(x_1, x_2) = \tau(\cos\theta, \sin\theta) + \rho(-\sin\theta, \cos\theta).$$

Hence, the variables (x_1, x_2) and (τ, ρ), for fixed θ, are related by the equations

$$x_1 = \tau\cos\theta - \rho\sin\theta, \quad x_2 = \tau\sin\theta + \rho\cos\theta. \tag{7.1}$$

Thus the Radon transform of the function $f(x_1, x_2)$, which we will denote by $\hat{f}(\rho, \theta)$, is defined by

$$\hat{f}(\rho, \theta) = \int_{-\infty}^{\infty} f(\tau\cos\theta - \rho\sin\theta, \tau\sin\theta + \rho\cos\theta)d\tau, \quad \rho \in \mathbb{R}, \quad \theta \in (0, 2\pi). \tag{7.2}$$

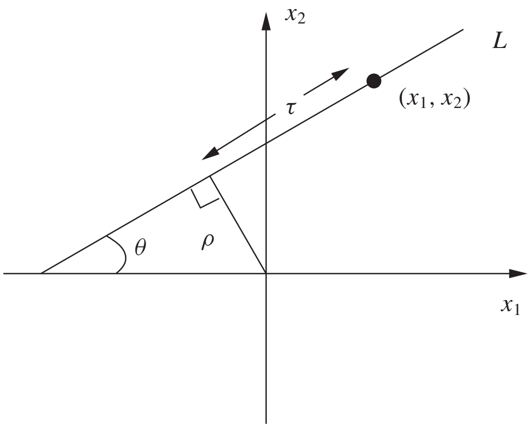


Figure 7.1. *Local coordinates for the mathematical formulation of PET and SPECT.*

In summary, the basic mathematical problem in CT is to reconstruct a function $f(x_1, x_2)$ from the knowledge of its Radon transform $\hat{f}(p, \theta)$.

The advent of CT made possible for the first time direct images of brain tissue. Furthermore, the subsequent development of magnetic resonance imaging (MRI) allowed striking discrimination between grey and white matter. This has had a tremendous impact on the entire field of medical imaging. Although the first applications of CT and MRI were in brain imaging, later these techniques were applied to many other areas of medicine. Indeed, it is impossible to think of medicine today without CT and MRI. However, in spite of their enormous impact these techniques are capable of imaging only *structural* as opposed to *functional* characteristics.

7.2 PET and SPECT

The study of functional characteristics became possible only in the late 1980s with the development of functional MRI, of positron emission tomography (PET), and of SPECT. Regarding the functional properties of the brain, using these new techniques it is now possible to observe with ever-increasing precision neural activity in living humans. For example, Figure 7.2 shows which parts of the brain are activated during a certain memory task. There exist, a vast number of clinical applications, including epilepsy, migraines, and differential diagnosis of schizophrenia and of Alzheimer's disease. Furthermore, just like with CT and MRI, the above new techniques are now used beyond neuroscience in a wide range of medical areas. These include pharmacology, oncology, and cardiology.

In PET, the patient is injected with a dose of flurodeoxyglucose (FGD) which is a normal molecule of glucose attached to an atom of radioactive fluorine. The fluorine atom in FDG suffers a radioactive decay emitting a positron, which when colliding with an electron liberates energy in the form of *two* beams of gamma rays which are picked up by the PET scanner simultaneously. In SPECT the situation is similar but instead of FDG one uses Xenon-133 which emits a *single* photon [88]. The more active cells consume more

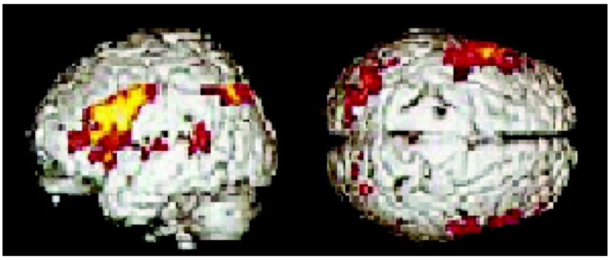


Figure 7.2. *Pet images during a memory task.*

glucose, and thus the measurement of the radioactive material provides an indirect measure of activation.

Let f, g, $L(x)$, denote the X-ray attenuation coefficient, the distribution of the radioactive material, and the part of the ray from the tissue to the detector. Then in SPECT the following integral I is known from the measurements:

$$I = \int_L e^{-\int_{L(x)} f ds} g d\tau.$$

This integral is called the attenuated (with respect to f) Radon transform of g and will be denoted by $\hat{g}_f$:

$$\hat{g}_f(\rho, \theta) = \int_{-\infty}^{\infty} e^{-\int_\tau^\infty f(s\cos\theta - \rho\sin\theta, s\sin\theta + \rho\cos\theta)ds} \\ \times g(\tau\cos\theta - \rho\sin\theta, \tau\sin\theta + \rho\cos\theta)d\tau, \quad \rho \in \mathbb{R}, \quad \theta \in (0, 2\pi). \tag{7.3}$$

Thus the basic mathematical problem of SPECT is to reconstruct the function $g(x_1, x_2)$ from the knowledge of its attenuated Radon transform $\hat{g}_f$ and of the associated X-ray attenuation coefficient $f(x_1, x_2)$.

7.3 The Mathematics of PET and SPECT

The author and Novikov derived in [19] the Radon transform by performing the spectral analysis of the following eigenvalue equation:

$$\frac{1}{2}\left(k + \frac{1}{k}\right)\frac{\partial \mu}{\partial x_1} + \frac{1}{2i}\left(k - \frac{1}{k}\right)\frac{\partial \mu}{\partial x_2} = f(x_1, x_2), \quad -\infty < x_1, x_2 < \infty, \quad k \in \mathbb{C}. \tag{7.4}$$

Although the Radon transform can be derived in a much simpler way by using the two-dimensional Fourier transform, the advantage of the derivation of [19] was demonstrated later by Novikov [18], who showed that the inverse of the attenuated Radon transform can be derived by applying a similar analysis to the following slight generalization of equation (7.4):

$$\frac{1}{2}\left(k + \frac{1}{k}\right)\frac{\partial \mu}{\partial x_1} + \frac{1}{2i}\left(k - \frac{1}{k}\right)\frac{\partial \mu}{\partial x_2} - f(x_1, x_2)\mu = g(x_1, x_2), \quad -\infty < x < \infty, \quad k \in \mathbb{C}. \tag{7.5}$$

It has been recently shown [89] that by scrutinizing the analysis of [18], it is possible to derive the attenuated Radon transform almost immediately. In this respect, we first review the main steps in the spectral analysis of (7.4). The derivation of the attenuated Radon transform will be a simple corollary of this analysis.

Proposition 7.1. Define the Radon transform $\hat{f}(\rho, \theta)$ of the function $f(x_1, x_2) \in S(\mathbb{R}^2)$ by (7.2). Then for all $(x_1, x_2) \in \mathbb{R}^2$,

$$f(x_1, x_2) = \frac{1}{4\pi} \left(\partial_{x_1} - i\partial_{x_2}\right) \int_0^{2\pi} e^{i\theta} J(x_1, x_2, \theta) d\theta, \tag{7.6}$$

where J is defined in terms of $\hat{f}$ by

$$J(x_1, x_2, \theta) = \frac{1}{i\pi} \fint_{-\infty}^{\infty} \frac{\hat{f}(\rho, \theta) d\rho}{\rho - (x_2 \cos\theta - x_1 \sin\theta)}, \quad \theta \in (0, 2\pi), \tag{7.7}$$

and $\fint$ denotes principal value integral.

Proof. We will derive the Radon transform pair by performing the spectral analysis of the eigenvalue equation (7.4). In order to solve the direct problem we first simplify equation (7.4) by introducing a change of variables from (x_1, x_2) to $(z, \bar{z})$, where

$$z = \frac{1}{2i}\left(k - \frac{1}{k}\right) x_1 - \frac{1}{2}\left(k + \frac{1}{k}\right) x_2,$$

$$\bar{z} = -\frac{1}{2i}\left(\bar{k} - \frac{1}{\bar{k}}\right) x_1 - \frac{1}{2}\left(\bar{k} + \frac{1}{\bar{k}}\right) x_2, \quad (x_1, x_2) \in \mathbb{R}^2, \quad k \in \mathbb{C}. \tag{7.8}$$

Using

$$\partial_{x_1} = \frac{1}{2i}\left(k - \frac{1}{k}\right)\partial_z - \frac{1}{2i}\left(\bar{k} - \frac{1}{\bar{k}}\right)\partial_{\bar{z}},$$

$$\partial_{x_2} = -\frac{1}{2}\left(k + \frac{1}{k}\right)\partial_z - \frac{1}{2}\left(\bar{k} + \frac{1}{\bar{k}}\right)\partial_{\bar{z}},$$

(7.4) becomes

$$\nu(|k|)\frac{\partial \mu}{\partial \bar{z}}(x_1, x_2, k) = f(x_1, x_2), \quad (x_1, x_2) \in \mathbb{R}^2, \quad k \in \mathbb{C}, \tag{7.9}$$

where

$$\nu(|k|) = \frac{1}{2i}\left(\frac{1}{|k|^2} - |k|^2\right). \tag{7.10}$$

We supplement (7.9) with the boundary condition

$$\mu = O\left(\frac{1}{z}\right), \quad z \to \infty. \tag{7.11}$$

Using the Pompieu (or $\bar{\partial}$, or Cauchy–Green) formula, (7.9) and (7.11) imply that for all $(x_1, x_2) \in \mathbb{R}^2$,

$$\mu = \frac{1}{2\pi i} \int\int_{\mathbb{R}^2} \frac{f(x_1', x_2')}{\nu(|k|)} \frac{dz' \wedge d\bar{z}'}{z' - z}, \quad k \in \mathbb{C}, \quad |k| \neq 1.$$

Hence, using

$$dz \wedge d\bar{z} = \frac{1}{2i} \left| |k|^2 - \frac{1}{|k|^2} \right| dx_1 dx_2$$

it follows that for all $(x_1, x_2) \in \mathbb{R}^2$ and $k \in \mathbb{C}$, $|k| \neq 1$, μ satisfies

$$\mu(x_1, x_2, k) = \frac{1}{2\pi i} \operatorname{sgn}\left(\frac{1}{|k|^2} - |k|^2\right) \int\int_{\mathbb{R}^2} f(x_1', x_2') \frac{dx_1' dx_2'}{z' - z}. \tag{7.12}$$

If k is either inside or outside the unit circle, the only dependence of μ on k is through z' and z, and thus μ is a sectionally analytic function with a "jump" across the unit circle of the complex k-plane. Equation (7.12) provides the solution of the direct problem.

In order to solve the inverse problem, we will formulate an RH problem in the complex k-plane. In this respect we note that (7.12) implies

$$\mu = O\left(\frac{1}{k}\right), \quad k \to \infty. \tag{7.13a}$$

Furthermore, we will show that for all $(x_1, x_2) \in \mathbb{R}^2$, μ satisfies the following "jump" condition:

$$\mu^+ - \mu^- = -\frac{1}{i\pi}(H\hat{f})(\rho, \theta), \quad \rho \in \mathbb{R}, \quad \theta \in (0, 2\pi), \tag{7.13b}$$

where H denotes the Hilbert transform in the variance ρ. This equation is a direct consequence of the following equations: Let μ^+ and μ^- denote the limits of μ as k approaches the unit circle from inside and outside, i.e.,

$$\mu^\pm \doteqdot \lim_{\varepsilon \to 0} \mu\left(x_1, x_2, (1 \mp \varepsilon)e^{i\theta}\right). \tag{7.14}$$

Then for all $(x_1, x_2) \in \mathbb{R}$,

$$\mu^\pm = \mp(P^\mp \hat{f})(\rho, \theta) - \int_\tau^\infty F(\rho, s, \theta) ds, \quad (\rho, \theta) \in \mathbb{R}^2, \quad \theta \in (0, 2\pi), \tag{7.15$^\pm$}$$

where $P^\pm$ denote the usual projectors in the variable ρ, i.e.,

$$(P^\pm \hat{f})(\rho) = \pm\frac{f(\rho)}{2} + \frac{1}{2i\pi}(Hf)(\rho); \quad (Hf)(\rho) = ⨍_{-\infty}^{\infty} \frac{f(\rho')}{\rho' - \rho} d\rho \tag{7.16}$$

and F denotes f in the coordinates (ρ, τ, θ), i.e.,

$$F(\rho, \tau, \theta) = f(\tau\cos\theta - \rho\sin\theta, \tau\sin\theta + \rho\cos\theta). \tag{7.17}$$

Indeed, in order to derive $(7.15)^\pm$ we note that the definition of z implies

$$(z - z') = \frac{1}{2i}\left(k - \frac{1}{k}\right)(x_1 - x_1') - \frac{1}{2}\left(k + \frac{1}{k}\right)(x_2 - x_2').$$

Let

$$k^{+} = (1-\varepsilon)e^{i\theta}, \quad k^{-} = (1+\varepsilon)e^{i\theta}, \quad \theta \in (0, 2\pi), \quad \varepsilon > 0. \tag{7.18}$$

Thus

$$\left(k^{+} \mp \frac{1}{k^{+}}\right) = (1-\varepsilon)e^{i\theta} \mp (1+\varepsilon)e^{-i\theta} + O(\varepsilon^2)$$

and similarly for $(k^{-} \mp 1/k^{-})$. Hence, for $\mu^{\pm}$, $z' - z$ is given by

$$\begin{aligned} z' - z = {} & (x_1' - x_1)\sin\theta - (x_2' - x_2)\cos\theta \\ & \pm i\varepsilon\left[(x_1' - x_1)\cos\theta + (x_2' - x_2)\sin\theta\right] + O(\varepsilon^2). \end{aligned} \tag{7.19}$$

Solving (7.1) for (ρ, τ) in terms of (x_1, x_2) we find

$$\tau = x_2 \sin\theta + x_1 \cos\theta, \qquad \rho = x_2\cos\theta - x_1\sin\theta. \tag{7.20}$$

The Jacobian of this transformation equals 1, hence $dx_1 dx_2 = d\rho d\tau$. Thus, replacing $z - z'$ in (7.12) by the RHS of equation (7.19) we find

$$\mu^{\pm} = \mp\frac{1}{2i\pi}\lim_{\varepsilon\to 0}\int\int_{\mathbb{R}^2}\frac{F(\rho', \tau', \theta)d\rho' d\tau'}{\rho' - [\rho \pm i\varepsilon(\tau' - \tau)]}. \tag{7.21}$$

In order to control this limit we must control the sign of $\tau' - \tau$. This suggests splitting the integral over $d\tau'$ as shown below,

$$\mu^{\pm} = \mp\frac{1}{2i\pi}\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\tau}\frac{F d\tau'}{\rho' - [\rho \pm i\varepsilon(\tau' - \tau)]} + \int_{\tau}^{\infty}\frac{F d\tau'}{\rho' - [\rho \pm i\varepsilon(\tau' - \tau)]}\right\}d\rho'.$$

In the first and second integrals above, $\tau' - \tau$ is negative and positive, respectively, and hence

$$\begin{aligned} \mu^{\pm} = {} & \mp\frac{1}{2i\pi}\int_{-\infty}^{\tau}\left\{\mp i\pi F(\rho, \tau', \theta) + (HF)(\rho, \tau', \theta)\right\}d\tau' \\ & \mp\frac{1}{2i\pi}\int_{\tau}^{\infty}\left\{\pm i\pi F(\rho, \tau', \theta) + (HF)(\rho, \tau', \theta)\right\}d\tau'. \end{aligned}$$

Adding and subtracting in the RHS of this equation the expression

$$\mp\frac{1}{2i\pi}\int_{\tau}^{\infty}(\mp)i\pi F(\rho, \tau', \theta)d\tau',$$

we find $(7.15)^{\pm}$.

The sectionally analytic function μ satisfies the estimate (7.13a) and has a jump across the unit circle, and thus for all $(x_1, x_2) \in \mathbb{R}^2$, it admits the following representation:

$$\mu = \frac{1}{2i\pi}\int_0^{2\pi}\frac{(\mu^{+} - \mu^{-})(\rho, \theta')ie^{i\theta'}d\theta'}{e^{i\theta'} - k}, \quad k \in \mathbb{C}, \quad |k| \neq 1, \quad \rho \in \mathbb{R}. \tag{7.22}$$

Replacing $\mu^{+} - \mu^{-}$ in this equation by the RHS of (7.13b) we find the following expression valid for all $(x_1, x_2) \in \mathbb{R}^2$:

$$\mu = -\frac{1}{2i\pi^2}\int_0^{2\pi}\frac{e^{i\theta'}(H\hat{f})(\rho, \theta')d\theta'}{e^{i\theta'} - k}, \quad k \in \mathbb{C}, \quad |k| \neq 1, \quad \rho \in \mathbb{R}. \tag{7.23}$$

This expression provides the solution of the inverse problem.

Using (7.4) and (7.23) it is straightforward to express f in terms of $\hat{f}$. One way of achieving this is to replace μ in (7.4) by the RHS of (7.23). A simpler alternative way is to compute the large k behavior of μ: Equation (7.23) implies

$$\mu \sim \left\{ \frac{1}{2i\pi^2} \int_0^{2\pi} e^{i\theta} (H\hat{f})(\rho, \theta) d\theta \right\} \frac{1}{k} + O\left(\frac{1}{k^2}\right), \quad k \to \infty.$$

Substituting this expression in (7.4) we find that the $O(1)$ term of (7.4) yields

$$f = \frac{1}{4i\pi^2} \left(\partial_{x_1} + \frac{1}{i} \partial_{x_2} \right) \int_0^{2\pi} e^{i\theta} (H\hat{f})(\rho, \theta) d\theta,$$

which is (7.6). □

Corollary 7.1. Let $k^{\pm}$ denote the limiting values of $k \in \mathbb{C}$ as it approaches the unit circle in the complex k-plane from inside and outside the unit circle; see (7.18). Let z be defined in terms of $(x_1, x_2) \in \mathbb{R}^2$ and $k \in \mathbb{C}$ by (7.8) and let $\nu(|k|)$ be defined by (7.10). Then

$$\lim_{k \to k^{\pm}} \left\{ \partial_{\bar{z}}^{-1} \left(\frac{f(x_1, x_2)}{\nu(|k|)} \right) \right\} = \mp (P^{\mp} \hat{f})(\rho, \theta) - \int_{\tau}^{\infty} F(\rho, s, \theta) ds,$$

$$(\rho, \tau) \in \mathbb{R}^2, \quad \theta \in (0, 2\pi), \tag{7.24}$$

where $\hat{f}$ is the Radon transform of f (see (7.2)), $P^{\pm}$ are the usual projectors in the variable ρ (see (7.16)), (ρ, τ) are defined in terms of (x_1, x_2) by (7.20), and F denotes f in the variables (ρ, τ, θ) (see (7.17)).

Proof. Equation (7.24) is a direct consequence of (7.9) and $(7.15)^{\pm}$. □

It turns out that the derivation of the attenuated Radon transform pair is a direct consequence of (7.5) and of the above corollary.

Proposition 7.2. Define the attenuated Radon transform $\hat{g}_f(\rho, \theta)$ of the function $g(x_1, x_2) \in S(\mathbb{R}^2)$ by (7.3), where $f(x_1, x_2) \in S(\mathbb{R}^2)$. Then

$$g(x_1, x_2) = \frac{1}{4\pi} \left(\partial_{x_1} - i \partial_{x_2} \right) \int_0^{2\pi} e^{i\theta} J(\rho, \tau, \theta) d\theta, \tag{7.25}$$

where (ρ, τ) are given in terms of (x_1, x_2) by (7.20) and J is defined in terms of $\hat{g}_f$ and f by

$$J(\rho, \tau, \theta) = -e^{\int_{\tau}^{\infty} f(s\cos\theta - \rho\sin\theta, s\sin\theta + \rho\cos\theta) ds}$$
$$\times \left[e^{P^- \hat{f}(\rho,\theta)} P^- e^{-P^- \hat{f}(\rho,\theta_+)} + e^{-P^+ \hat{f}(\rho,\theta)} P^+ e^{P^- \hat{f}(\rho,\theta)} \right] \hat{g}_f(\rho, \theta),$$

$$(\rho, \tau) \in \mathbb{R}^2, \quad \theta \in (0, 2\pi). \tag{7.26}$$

Proof. Equation (7.5) can be rewritten in the form

$$\frac{\partial \mu}{\partial \bar{z}} + \frac{f}{\nu}\mu = \frac{g}{\nu}.$$

Hence

$$\frac{\partial}{\partial \bar{z}}\left[\mu e^{\partial_{\bar{z}}^{-1}\left(\frac{f}{\nu}\right)}\right] = \frac{g}{\nu} e^{\partial_{\bar{z}}^{-1}\left(\frac{f}{\nu}\right)}$$

or

$$\mu e^{\partial_{\bar{z}}^{-1}\left(\frac{f}{\nu}\right)} = \partial_{\bar{z}}^{-1}\left[\left(\frac{g}{\nu}\right) e^{\partial_{\bar{z}}^{-1}\left(\frac{f}{\nu}\right)}\right], \quad (x_1, x_2) \in \mathbb{R}^2, \quad k \in \mathbb{C}. \tag{7.27}$$

This equation provides the solution of the direct problem, i.e., it defines a sectionally analytic function μ with the estimate (7.13a), which has a jump across the unit circle of the complex k-plane. Hence, μ is given by (7.22). All that remains is to determine the jump $\mu^+ - \mu^-$. This involves computing the limits as $k \to k^\pm$ of $\partial_{\bar{z}}^{-1}(f/\nu)$, and thus it can be achieved using (7.24). Equation (7.27) implies

$$\mu^\pm e^{\mp P^\mp \hat{f}} e^{-\int_\tau^\infty F(\rho,s,\theta)ds} = \lim_{k\to k^\pm} \partial_{\bar{z}}^{-1}\left\{\frac{g}{\nu} e^{\mp P^\mp \hat{f}} e^{-\int_\tau^\infty F(\rho,s,\theta)ds}\right\}. \tag{7.28}$$

For the computation of the RHS of this equation we use again (7.24), where f is now replaced by g times the two exponentials appearing in {} of (7.28). Hence the RHS of (7.28) yields

$$\mp P^\mp e^{\mp P^\mp \hat{f}} \hat{g}_f - \int_\tau^\infty G(\rho, \tau', \theta) e^{\mp P^\mp \hat{f}} e^{-\int_{\tau'}^\infty F(\rho,s,\theta)ds} d\tau',$$

where G denotes g in the coordinates (ρ, τ, θ). The term $\exp[\mp P^\mp \hat{f}]$ is independent of τ', and hence this term comes out of the integral $\int_\tau^\infty$, and furthermore the same terms appear in the LHS of (7.28). Hence,

$$\mu^+ - \mu^- = -J,$$

where J is defined in (7.26). Then (7.22) yields

$$\mu = -\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta'} J(\rho, \tau, \theta')d\theta'}{e^{i\theta'} - k}.$$

Hence,

$$\mu = \left\{\frac{1}{2\pi}\int_0^{2\pi} e^{i\theta} J(\rho, \tau, \theta)d\theta\right\}\frac{1}{k} + O\left(\frac{1}{k^2}\right), \quad k \to \infty.$$

Substituting this expression in (7.5) we find that the $O(1)$ term of (7.5) yields (7.25). ∎

7.4 Numerical Implementation

The numerical implementation of the inverse attenuated Radon transform, i.e., of (7.25), using either cubic splines or Chebyshev approximations is presented in [89]. A typical implementation using a technique based on the fast Fourier transform is shown in image (c) of Figure 7.3. The images (b), (c), (d) depict the reconstructions of a realistic cardiac phantom depicted in Figure 7.3(a), using three different techniques. The reconstruction

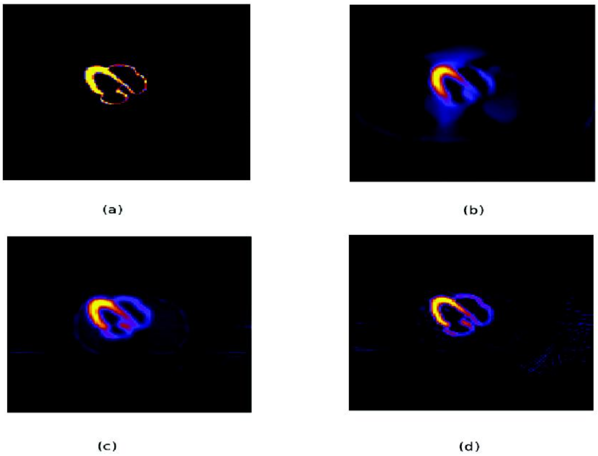


Figure 7.3. *Different reconstructions of a cardiac phantom.*

(b) uses the approximation of $f = 0$, which reduces the attenuated Radon transform to the classical Radon transform; the reconstruction of the latter transform uses a technique based on the fast Fourier transform, which is called filter back projection (this is actually what is used now for SPECT in most hospitals). The reconstruction in (d) uses an improved mathematical model for SPECT, which takes into account the fact that the collimator actually receives "cones" instead of rays. This leads to a modified attenuated Radon transform which can also be inverted analytically [90]. The incorporation of noise into these analytical algorithms is a challenging problem which is under investigation.

Chapter 8

The Dirichlet to Neumann Map for a Moving Boundary

It was shown in section 1.4 that the characterization of the Dirichlet to Neumann map for the heat equation on the half-line is based on the analysis of the global relation and on the inversion of the following integral:

$$\hat{f}(k) = \int_0^T e^{k^2 s} f(s) ds, \quad T > 0, \quad k \in \mathbb{C}. \tag{8.1}$$

It turns out that the characterization of the analogous map for the heat equation on the moving boundary $\{l(t) < x < \infty, 0 < t < T\}$ requires the inversion of the integral

$$\hat{f}(k) = \int_0^T e^{k^2 s - ikl(s)} f(s) ds, \quad T > 0, \quad k \in \mathbb{C}. \tag{8.2}$$

The integral (8.1) can be inverted in an elementary manner using the Fourier transform. Alternatively, it can be inverted using the spectral analysis of the following eigenvalue equation for the function $\mu(t, k)$ (compare with Example 6.2 of Chapter 6):

$$\mu_t + k^2 \mu = kf(t), \quad 0 < t < T, \quad k \in \mathbb{C}. \tag{8.3}$$

The integral (8.2) apparently *cannot* be inverted using the Fourier transform. However, it can be inverted using the spectral analysis of the following eigenvalue equation for the function $\mu(t, k)$:

$$\mu_t + (k^2 - ik\dot{l}(t))\mu = kf(t), \quad 0 < t < T, \quad k \in \mathbb{C}. \tag{8.4}$$

There exists an important difference between (8.3) and (8.4): Equation (8.3) admits a sectionally analytic solution $\mu(t, k)$ in k, but *no* such solution exists for (8.4). Hence, the solution of the inverse problem associated with equation (8.4) requires the formulation of a $\bar{\partial}$-problem. A general procedure for deriving such $\bar{\partial}$-problems was introduced in the works of Pelloni and the author [49]. However, in these papers the inversion formula for $f(t)$ was left in terms of a two-dimensional integral, and thus it did *not* provide an effective way of constructing $f(t)$. It was later shown in [20] that the relevant double integral can

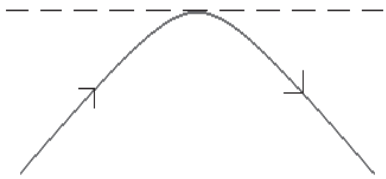


Figure 8.1. *The curve* $\Gamma(t)$.

be expressed in terms of single integrals, which in turn yield a Volterra integral equation for $f(t)$. This equation involves a kernel with a strong decay which in particular yields effective numerical computations [21]. Using the general methodology introduced in [20], the following result is obtained in [21].

Proposition 8.1. Let $\hat{f}(k)$ be defined in terms of $f(t)$ by (8.2) where $l(t)$ is a smooth function satisfying

$$\ddot{l}(t) > 0, \qquad 0 < t < T; \qquad l(0) = 0. \tag{8.5}$$

Then $f(t)$ can be expressed in terms of $\hat{f}(k)$ through the solution of the Volterra integral equation

$$\frac{3}{4}f(t) = -\frac{1}{2\pi i}\int_{\Gamma(t)} ke^{-k^2t+ikl(t)}\hat{f}(k)dk + \int_0^t f(s)K(s,t)ds, \quad 0 < t < T, \tag{8.6}$$

where the curve $\Gamma(t)$, depicted in Figure 8.1, is defined by the equation

$$\Gamma(t) = \left\{k \in \mathbb{C},\ k_R^2 - k_I^2 + k_I\dot{l}(t) = 0;\ -\infty < k_R < \infty,\ k_I < 0;\ 0 < t < T\right\} \tag{8.7a}$$

and $K(s,t)$ is defined by the following equations:

$$\begin{aligned} K(s,t) = -\frac{1}{2\pi}\int_0^\infty \Bigg\{ &\left[1 - \frac{i}{2}\left(\frac{\nu}{\sqrt{\nu^2 - \dot{l}(s)\nu}} + \frac{\sqrt{\nu^2 - \dot{l}(s)\nu}}{\nu}\right)\right] B_+(\nu,s,t) \\ &- \left[1 + \frac{i}{2}\left(\frac{\nu}{\sqrt{\nu^2 - \dot{l}(s)\nu}} + \frac{\sqrt{\nu^2 - \dot{l}(s)\nu}}{\nu}\right)\right] B_-(\nu,s,t) \Bigg\} d\nu, \end{aligned} \tag{8.7b}$$

$$\begin{aligned} B_\pm(\nu,s,t) = &\left(\pm\sqrt{\nu^2 - \dot{l}(s)\nu} + i\nu\right) \\ &\times \exp\left\{-\nu\left[\dot{l}(s) - \vartheta(t,s)\right](s-t) \pm i\sqrt{\nu^2 - \dot{l}(s)\nu}\,[2\nu - \vartheta(t,s)](s-t)\right\}, \end{aligned} \tag{8.7c}$$

$$\vartheta(t,s) = \frac{l(t) - l(s)}{t - s}.$$

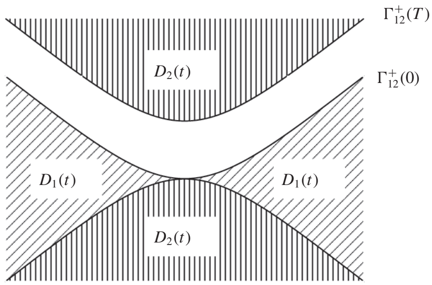


Figure 8.2. *The domains $D_1(t)$ and $D_2(t)$.*

Proof. In order to solve the direct problem, we first define the following t-dependent domains in the complex k-plane:

$$D_1(t) = \left\{k_I > 0,\ k_R^2 - k_I^2 + k_I \dot{l}(0) > 0\right\} \cup \left\{k_I < 0,\ k_R^2 - k_I^2 + k_I \dot{l}(t) > 0\right\}, \quad (8.8a)$$

$$D_2(t) = \left\{k_I > 0,\ k_R^2 - k_I^2 + k_I \dot{l}(T) < 0\right\} \cup \left\{k_I < 0,\ k_R^2 - k_I^2 + k_I \dot{l}(t) < 0\right\}. \quad (8.8b)$$

These domains are depicted in Figure 8.2, where the curves $\Gamma_{12}^{+}(0)$ and $\Gamma_{12}^{+}(T)$ are the curves obtained by letting $t = 0$ and $t = T$ in the curve defined by

$$\Gamma_{12}^{+}(t) = \left\{k_R^2 - k_I^2 + k_I \dot{l}(t) = 0,\ k_I > 0\right\}. \quad (8.9)$$

A solution $\mu(t,k)$ of (8.4) bounded for all complex k is given by

$$\mu(t,k) = \mu_j(t,k), \quad k \in D_j(t), \quad 0 < t < T, \quad j = 1,2,3, \quad (8.10)$$

where $D_3(t)$ is the complement of $\{D_1(t) \cup D_2(t)\}$ in the entire complex k-plane and the functions μ_j, $j = 1,2,3$, are defined as follows:

$$\mu_j(t,k) = k \int_{t_j}^{t} e^{k^2(s-t) - ik[l(s)-l(t)]} f(s)ds,\ 0 < t < T,\ k \in D_j,\ j = 1,2,3, \quad (8.11)$$

$$t_1 = 0, \qquad t_2 = T, \qquad t_3 = S(k_R, k_I), \quad (8.12)$$

where $S(k_R, k_I)$ is the unique solution for t of the equation

$$k_R^2 - k_I^2 + k_I \dot{l}(t) = 0,\ 0 < t < T,\ k_I > 0,\ -\infty < k_R < \infty : t = S(k_R, k_I). \quad (8.13)$$

Indeed the functions μ_1 and μ_2 are entire functions of k, which are bounded as $k \to \infty$ in the domains $D_1(t)$ and $D_2(t)$, respectively. These domains are determined by the real part of the exponential appearing in (8.11), namely by

$$\exp\left\{(s-t)\left[k_R^2 - k_I^2 + k_I \frac{(l(s) - l(t))}{(s-t)}\right]\right\} = \exp\left\{(s-t)\left[k_R^2 - k_I^2 + k_I \dot{l}(\tau)\right]\right\},$$

where τ is in the interval bounded by t and s. For the function μ_1, $s \le t$, and thus μ_1 is bounded if and only if

$$k_R^2 - k_I^2 + k_I \dot{l}(\tau) \ge 0 \qquad \forall \tau \in [0,t],$$

which, taking into consideration that $\dot{l}$ is an increasing function, characterizes $D_1(t)$. For μ_2, $s \geq t$, and thus μ_2 is bounded if and only if

$$k_R^2 - k_I^2 + k_I \dot{l}(\tau) \leq 0 \qquad \forall \tau \in [t, T],$$

which characterizes $D_2(t)$.

In order to prove that μ_3 is bounded in $D_3(t)$ we distinguish two cases.

- $0 \leq S(k_R, k_I) < t$.

  In this case $s < t$; thus we need to prove that $k_R^2 - k_I^2 + k_I \dot{l}(\tau) \geq 0$, which follows from the facts that $\dot{l}(\tau) > \dot{l}(s)$, $k_I > 0$, and $k_I \dot{l}(s) = k_I^2 - k_R^2$.

- $t < S(k_R, k_I) \leq T$.

  The situation is similar to the first case, but $s - t \geq 0$ and $\dot{l}(\tau) < \dot{l}(s)$.

Integration by parts of the equations defining the μ_j's implies the following asymptotic behavior:

$$\mu_j = O\left(\frac{1}{k}\right), \quad k \to \infty, \quad k \in D_j, \quad j = 1, 2, 3. \tag{8.14}$$

In order to solve the inverse problem, we note that the function $\mu(t, k)$ defined by (8.10)–(8.12) is bounded in the entire complex plane including infinity. Hence, it is possible to find an alternative representation for μ using the Pompieu, Cauchy–Green or d-bar formula, i.e., (5.17),

$$\mu(t, k, \bar{k}) = \frac{1}{2\pi i} \int_{\Gamma(t)} \frac{\text{“jump”}}{\lambda - k} d\lambda - \frac{1}{\pi} \iint_{D(t)} \frac{\partial \mu(t, \lambda, \bar{\lambda})}{\partial \lambda} \frac{d\lambda_R d\lambda_I}{\lambda - k}, \tag{8.15}$$

$$0 < t < T, \quad k \in \mathbb{C},$$

where Γ denotes the contour along which the function μ has a “jump” discontinuity and D is the domain in which $\partial\mu/\partial\bar{k} \neq 0$. If $k = k_R + i k_I$, then

$$\frac{\partial}{\partial \bar{k}} = \frac{1}{2}\left(\frac{\partial}{\partial k_R} + i \frac{\partial}{\partial k_I}\right). \tag{8.16}$$

For the function μ defined by (8.10)–(8.12), μ_1 and μ_2 are analytic in k; and thus $\partial\mu/\partial\bar{k} = \partial\mu_3/\partial\bar{k}$ and $D = D_3(t)$. Furthermore, there are *no* jumps between μ_3 and μ_2, and between μ_3 and μ_1. Indeed, the intersection of $D_3(t)$ and $D_1(t)$ occurs on the curve $\Gamma_{12}^+(T)$, and on this curve $S(k_R, k_I) = T$; thus $\mu_3 = \mu_2$. Similarly, the intersection of D_3 with D_1 occurs on the curve $\Gamma_{12}^+(0)$, and on this curve $S(k_R, k_I) = 0$; thus $\mu_3 = \mu_1$. Therefore, the only jump occurs on the curve $\Gamma(t)$ (which is the intersection of D_1 and D_2) and this jump equals

$$\mu_1(t, k) - \mu_2(t, k) = k e^{-k^2 t + ikl(t)} \hat{f}(k), \qquad k \in \Gamma(t), \qquad 0 < t < T.$$

In summary, if $\mu(t, k)$ is defined by (8.10)–(8.12), then μ also admits the following representation for $k \in \mathbb{C} \backslash \Gamma(t)$ and $0 < t < T$:

$$\mu(t, k, \bar{k}) = \frac{1}{2\pi i} \int_{\Gamma(t)} \lambda e^{-\lambda^2 t + i\lambda l(t)} \hat{f}(\lambda) \frac{d\lambda}{\lambda - k} - \frac{1}{\pi} \iint_{D_3(t)} \frac{\partial \mu_3(t, \lambda, \bar{\lambda})}{\partial \bar{\lambda}} \frac{d\lambda_R d\lambda_I}{\lambda - k}, \tag{8.17}$$

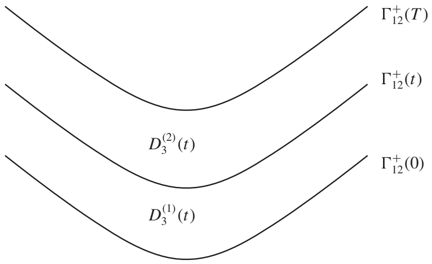


Figure 8.3. *The domains $D_3^{(1)}(t)$ and $D_3^{(2)}(t)$.*

where μ_3 is defined by (8.11) with $j = 3$.

The representation of the function μ defined by (8.17) involves $\hat{f}(k)$ and $\partial\mu_3/\partial\bar{k}$. Thus, there exists a relation between $f(t)$ and $\{\hat{f}(k), \partial\mu_3/\partial\bar{k}\}$. In order to obtain this relation, rather than replacing μ in (8.4) by the RHS of (8.17), we consider the large k behavior of $\mu(t,k,\bar{k})$. Equation (8.17) implies

$$\mu(t,k,\bar{k}) = \frac{\mu_0(t)}{k} + O\left(\frac{1}{k^2}\right), \quad k \to \infty, \quad \mu_0 = \lim_{k\to\infty}(k\mu).$$

Substituting the above expansion in the ODE (8.4), we find $\mu_0 = f(t)$, and hence for all $0 < t < T$,

$$f(t) = -\frac{1}{2\pi i}\int_{\Gamma(t)} k e^{-k^2 t + ikl(t)}\hat{f}(k)dk + \frac{1}{\pi}\iint_{D_3(t)} \frac{\partial\mu_3(t,k,\bar{k})}{\partial\bar{k}} dk_R dk_I. \tag{8.18}$$

In order to obtain a Volterra integral equation for $f(t)$ we must simplify the second term in the RHS of (8.18). Actually, we will show that this term equals

$$\frac{1}{4}f(t) + \int_0^t f(s)K(s,t)ds, \tag{8.19}$$

where the kernel $K(s,t)$ is defined by (8.7b).

Indeed, the curve $\Gamma_{12}^+(t)$ defined in (8.9) subdivides the domain $D_3(t)$ into the two subdomains $D_3^{(1)}(t)$ and $D_3^{(2)}(t)$; see Figure 8.3. The integral over the domain $D_3^{(2)}(t)$ equals $\frac{f}{4}$. Indeed, using the identity $2i dk_R dk_I = d\bar{k} \wedge dk$, as well as the complex form of Green's theorem in the domain $D_3^{(2)}(t)$, we find

$$2i\iint_{D_3^{(2)}(t)} \frac{\partial\mu_3}{\partial\bar{k}}(t,k,\bar{k})dk_R dk_I = \left(\int_{\Gamma_{12}^+(t)} + \int_{\Gamma_{12}^+(T)}\right)\mu_3(t,k,\bar{k})dk. \tag{8.20}$$

On the curve $\Gamma_{12}^+(t)$, $S(k_R,k_I) = t$, and thus $\mu_3 = 0$, whereas on the curve $\Gamma_{12}^+(T)$, $S(k_R,k_I) = T$. Hence the RHS of (8.20) equals

$$\int_{\Gamma_{12}^+(T)} k\left(\int_T^t e^{k^2(s-t)-ik[l(s)-l(t)]}f(s)ds\right)dk.$$

The integrand of this integral is bounded and analytic in the domain above the curve $\Gamma_{12}^{+}(T)$, and its large k behavior is given by $\frac{f(t)}{k}$. Thus, the Cauchy theorem in the domain above the curve $\Gamma_{12}^{+}(T)$ implies that the above integral equals

$$f(t)\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} i d\vartheta = if(t)\frac{\pi}{2}.$$

Hence the contribution of the second term on the RHS of (8.18) associated with the domain $D_3^{(2)}(t)$ equals $\frac{f(t)}{4}$.

In order to compute the integral over $D_3^{(1)}(t)$, we note that using (8.11) with $j = 3$, as well as equations (8.12) and (8.16), we find

$$\frac{\partial \mu_3}{\partial \bar{k}} = -\frac{1}{2}\left(\frac{\partial S}{\partial k_R} + i\frac{\partial S}{\partial k_I}\right) B(k, S, t) g_1(S), \tag{8.21a}$$

where

$$B(k, s, t) = ke^{k^2(s-t)-ik[l(s)-l(t)]}. \tag{8.21b}$$

Then we obtain the following contribution to the double integral in (8.18) associated with $D_3^{(1)}(t)$:

$$\begin{aligned} &\frac{1}{\pi}\iint_{D_3^{(1)}(t)} k\frac{\partial \mu_3}{\partial \bar{k}}(t, k, \bar{k}) dk_R dk_I \\ &= -\frac{1}{2\pi i}\iint_{D_3^{(1)}(t)} \left(\frac{\partial S}{\partial k_R} + i\frac{\partial S}{\partial k_I}\right) B(k, S, t), t) g_1(S) dk_R dk_I. \end{aligned} \tag{8.22}$$

We make the change of variables

$$(k_R, k_I) \longrightarrow (\lambda, \nu), \qquad \lambda = k_I - \frac{k_R^2}{k_I}, \qquad \nu = k_I,$$

which implies

$$D_3^{(1)}(t) = \left\{(\lambda, \nu) : \dot{l}(0) \le \lambda \le \dot{l}(t),\ 0 \le \nu < \infty\right\}, \quad S = S(\lambda).$$

Using $\nu = k_I$, $\lambda = k_I - k_R^2/k_I$, it follows that

$$\frac{\partial S}{\partial k_R} = -2\frac{k_R}{k_I}\frac{\partial S}{\partial \lambda}, \quad \frac{\partial S}{\partial k_I} = \left(1 + \frac{k_R^2}{k_I^2}\right)\frac{\partial S}{\partial \lambda}$$

and

$$d\nu d\lambda = -2\frac{k_R}{k_I} dk_I dk_R, \quad \frac{k_R}{k_I} = \frac{(\nu^2 - \nu\lambda)^{1/2}}{\nu}.$$

Thus the relevant integral in (8.22) becomes

$$\begin{aligned} &\frac{1}{\pi}\int\int_{D_3^{(1)}(t)} k\frac{\partial \mu_3}{\partial \bar{k}}(t, k, \bar{k}) dk_R dk_I \\ &= -\frac{1}{2\pi i}\int_0^\infty \int_{l'(0)}^{l'(t)} \left[1 - \frac{i}{2}\left(\frac{\nu}{\sqrt{\nu^2 - \lambda\nu}} + \frac{\sqrt{\nu^2 - \lambda\nu}}{\nu}\right)\right] \\ &\times \frac{dS}{d\lambda} B\left(\sqrt{\nu^2 - \lambda\nu} + i\nu, S(\lambda), t\right) g_1(S) d\lambda d\nu. \end{aligned} \tag{8.23}$$

Under the change of variable $\lambda \to S(\lambda) = s$, $0 \le s \le t$, the double integral in the RHS of (8.23) can be written as

$$-\frac{1}{2\pi i}\int_0^\infty \int_0^t \left\{ \left[1 - \frac{i}{2}\left(\frac{\nu}{\sqrt{\nu^2 - \dot{l}(s)\nu}} + \frac{\sqrt{\nu^2 - \dot{l}(s)\nu}}{\nu}\right)\right] B_+(\nu, s, t) \right. \\ \left. + \left[1 + \frac{i}{2}\left(\frac{\nu}{\sqrt{\nu^2 - \dot{l}(s)\nu}} + \frac{\sqrt{\nu^2 - \dot{l}(s)\nu}}{\nu}\right)\right] B_-(\nu, s, t) \right\} g_1(s) ds d\nu, \tag{8.24}$$

where $B_\pm$ are defined in (8.7c). In the above integrals the order of integration can be inverted and the resulting expressions are well defined; the expression in (8.24) is therefore equal to

$$\int_0^t g_1(s) K(s, t) ds$$

with the kernel $K(s, t)$ given by (8.7b). □

Remark 8.1. Using the mean value theorem it follows that $\vartheta(t, s) = \dot{l}(\tau)$, $s < \tau < t$. This implies that the real part of the exponential in the RHS of (8.7c) is negative; hence, the kernel $K(s, t)$ decays exponentially as $\nu \to \infty$.

Remark 8.2. Replacing μ in (8.4) with the RHS of (8.17), we find the expression for $f(t)$ given by the RHS of (8.18), as well as the equation

$$\frac{1}{2\pi i}\int_{\Gamma_{12}(t)} k e^{-k^2 t + ikl(t)} \hat{f}(k) dk - \frac{1}{\pi}\iint_{D_3(t)} k \frac{\partial \mu_3}{\partial \bar{k}}(t, k, \bar{k}) dk_R dk_I = i\dot{l}(t) f(t). \tag{8.25}$$

The LHS of this equation equals $\tilde{\mu}(t)$, where $\tilde{\mu}(t)$ is the coefficient of the $\frac{1}{k^2}$ term in the large k expansion of the function $\mu(t, k, \bar{k})$ defined by (8.17). Thus (8.25) can be rewritten in the form

$$\tilde{\mu}(t) = i\dot{l}(t) f(t),$$

which can also be obtained by substituting the large k expansion of $\mu(t, k, \bar{k})$ in the ODE (8.4).

Using the result of Proposition 8.1 and employing the global relation, it is straightforward to construct the Dirichlet to Neumann map for a moving boundary.

8.1 The Solution of the Global Relation

Proposition 8.2. Let $q(x, t)$ satisfy the heat equation (15a) in the domain Ω defined by

$$\Omega = \{0 < t < T, \quad l(t) < x < \infty\}, \tag{8.26}$$

where the function $l(t)$ satisfies the conditions (8.5). Let q satisfy

$$q(x, 0) = q_0(x), \quad 0 < x < \infty; \quad q(l(t), t) = g_0(t), \quad 0 < t < T, \tag{8.27}$$

where the functions $g_0(t)$, $q_0(x)$ have sufficient smoothness and $q_0(x)$ has sufficient decay for large x. Suppose that there exists a solution $q(x,t)$ with sufficient smoothness and decay. Then the function $q_x(l(t,t)) = g_1(t)$ satisfies the following Volterra integral equation for $0 < t < T$:

$$\frac{3}{4} g_1(t) = \frac{1}{2\sqrt{\pi}} \left[\frac{1}{\sqrt{t}} \int_0^\infty e^{-\frac{[l(t)-x]^2}{4t}} \dot{q}_0(x)dx - \int_0^t \frac{e^{-\frac{[l(t)-l(s)]^2}{4(t-s)}}}{\sqrt{t-s}} \dot{g}_0(s)ds \right] + \int_0^t g_1(s)K(s,t)ds, \tag{8.28}$$

where $\dot{q}_0$, $\dot{g}_0$ denote the derivatives of q_0, g_0 and the function $K(s,t)$, $0 < s < t < T$, is defined by (8.7b).

Proof. Suppose that there exists a solution $q(x,t)$ with sufficient smoothness in the closure $\overline{\Omega}$ of the domain Ω. Then, the application of Green's theorem in the domain Ω yields

$$\int_{\partial\Omega} e^{-ikx+k^2t}\{q(x,t)dx + [q_x(x,t) + ikq(x,t)]dt\} = 0, \quad k \in \mathbb{C}, \tag{8.29}$$

or

$$\int_0^T e^{k^2t-ikl(t)} \left\{q_x(l(t),t) + [\dot{l}(t) + ik]g_0(t)\right\} dt = \hat{q}_0(k) - e^{k^2T} \int_{l(T)}^\infty e^{-ikx} q(x,T)dx, \tag{8.30}$$

where $\partial\Omega$ denotes the boundary of Ω and $\hat{q}_0(k)$ denotes the Fourier transform of the initial condition $q_0(x)$. Equation (8.30) is valid only for $\operatorname{Im} k < 0$; this restriction is imposed in order for the function $\hat{q}_0(k)$ as well as the integral involving $q(x,T)$ to make sense. Equation (8.30) is the global relation. Equation (8.30) is of the form of the basic equation (8.2), where $f(s)$ is replaced by $g_1(s)$ and $\hat{f}(k)$ is replaced by the expression

$$\hat{q}_0(k) - \int_0^T e^{k^2s-ikl(s)}[\dot{l}(s) + ik]g_0(s)ds - e^{k^2T} \int_{l(T)}^\infty e^{-ikx} q(x,T)dx. \tag{8.31}$$

Furthermore the global relation is valid for $\operatorname{Im} k \leq 0$, and therefore it is valid for k on $\Gamma(t)$.

Replacing $\hat{f}(k)$ in the first term on the RHS of (8.6) by the expression in (8.31), we obtain three contributions. The contribution from the third term in the expression (8.31) vanishes. Indeed, this latter term yields the integrand

$$e^{k^2(T-t)+ik[l(t)-l(T)]} k \int_{l(T)}^\infty e^{-ik[x-l(T)]} q(x,T)dx. \tag{8.32}$$

The exponential multiplying the above integral is bounded in the domain below the curve $\Gamma(t)$, whereas the integral in (8.32) is bounded and analytic for $\operatorname{Im} k < 0$ and is of $O(\frac{1}{k})$ as $k \to \infty$. Thus, an application of Jordan's lemma in the variable l, where

$$l = -k^2; \quad k = |k|e^{-\frac{i\pi}{4}}, \quad 0 < |k| < \infty, \quad -\infty < l < 0,$$
$$k = |k|e^{\frac{3i\pi}{4}}, \quad 0 < |k| < \infty, \quad 0 < l < \infty,$$

implies that the integral of the expression (8.32) over $\Gamma(t)$ vanishes.

We will now show that the contribution of the first two terms of the expression in (8.31) yields the first two terms in the RHS of (8.28).

Multiplying these two terms by k, integrating by parts, and using $q_0(0) = g_0(0)$ we find

$$-i\left\{\int_0^\infty e^{-ikx}\dot{q}_0(x)dx + e^{k^2T-ikl(T)}g_0(T) - \int_0^T e^{k^2s-ikl(s)}\dot{g}_0(s)ds\right\}. \tag{8.33}$$

We split the integral $\int_0^T$ into the integrals $\int_0^t$ and $\int_t^T$. After multiplying the expression in (8.33) by $\exp[-k^2t + il(t)]$ and integrating over $\Gamma(t)$, it follows that the terms involving $g_0(T)$ and $\int_t^T$ vanish. Hence we find the expression

$$-i\int_0^\infty K_1(t,x)\dot{q}_0(x)dx + i\int_0^t K_2(t,s)\dot{g}_0(s)ds, \tag{8.34}$$

where

$$K_1(t,x) = \int_{\Gamma(t)} e^{-k^2t+ik[l(t)-x]}dk, \qquad t > 0, \qquad 0 < x < \infty, \tag{8.35}$$

and

$$K_2(t,s) = \int_{\Gamma(t)} e^{-k^2(t-s)+ik[l(t)-l(s)]}dk, \qquad 0 < t < s < T. \tag{8.36}$$

Using

$$-k^2t + ik[l(t)-x] = -\lambda_1^2 - \frac{[l(t)-x]^2}{4t}, \qquad \lambda_1 = k\sqrt{t} - \frac{i[l(t)-x]^2}{4t},$$

it follows that

$$K_1(t,x) = \frac{e^{-\frac{[l(t)-x]^2}{4t}}}{\sqrt{t}}\int_{\tilde{\Gamma}(t,x)} e^{-\lambda_1^2}d\lambda_1, \tag{8.37}$$

where $\tilde{\Gamma}(t,x)$ is the curve obtained from $\Gamma(t)$ under the transformation $k \to \lambda_1(k)$.

Similarly, using

$$-(t-s)k^2 + ik[l(t)-l(s)] = -\lambda_2^2 - \frac{[l(t)-l(s)]^2}{4(t-s)}, \qquad \lambda_2 = k\sqrt{t-s} - \frac{i[l(t)-l(s)]^2}{4(t-s)},$$

it follows that

$$K_2(t,s) = \frac{e^{-\frac{[l(t)-l(s)]^2}{4(t-s)}}}{\sqrt{t-s}}\int_{\hat{\Gamma}(t,s)} e^{-\lambda_2^2}d\lambda_2, \tag{8.38}$$

where $\hat{\Gamma}(t,s)$ is the curve obtained from $\Gamma(t)$ under the transformation $k \to \lambda_2(k)$.

Using the fact that $\exp(-\lambda^2)$ is bounded in $\arg\lambda \in \left\{[-\frac{\pi}{4}, \frac{\pi}{4}] \cap [\frac{3\pi}{4}, \frac{5\pi}{4}]\right\}$, it follows that both $\tilde{\Gamma}(t,x)$ and $\hat{\Gamma}(t,s)$ can be deformed to the real axis, and hence the two relevant integrals equal $\sqrt{\pi}$. Then the expression in (8.34) yields the first two terms in (8.28).

8.2 Examples

Numerical evaluations of the Neumann data $g_1(t)$ are presented in [21] for the case of $l(t) = \frac{t^2}{2}$ and the following two different choices of initial and boundary conditions:

$$\text{(a)} \quad q_0(x) = 0, \qquad g_0(t) = \sin(t), \tag{8.39}$$

$$\text{(b)} \quad q_0(x) = xe^{-x}, \qquad g_0(t) = 0. \tag{8.40}$$

In this case the function $B_+(\nu, s, t)$ takes the explicit form

$$\begin{aligned} B_+(\nu, s, t) &= \left(\sqrt{\nu^2 - s\nu} + i\nu\right) \\ &\times \exp\left\{-\frac{\nu}{2}(s-t)^2 + i\sqrt{\nu^2 - s\nu}(s-t)\left[2\nu - \frac{1}{2}(s+t)\right]\right\} \end{aligned} \tag{8.41}$$

and therefore it is obvious that the kernel $K(s,t)$ decays exponentially as $\nu \to \infty$.

For the first set of initial and boundary data, we observe that the first integral in the RHS of (8.28) is identically zero, while the second integral is given by

$$-\frac{1}{2\sqrt{\pi}} \int_0^t \frac{\cos(s)}{\sqrt{t-s}} \exp\left[-\frac{(t+s)(t^2-s^2)}{16}\right] ds. \tag{8.42}$$

For the second set of initial and boundary data, we find that the second term in the RHS of (8.28) is identically zero, while the first term equals

$$\frac{1}{2\sqrt{\pi t}} \int_0^\infty (1-x)e^{-x} e^{-\frac{(t^2-2x)^2}{16t}} dx. \tag{8.43}$$

The results of the numerical simulations are shown in Figures 8.4 and 8.5 for the two cases (8.39) and (8.40), respectively. In both cases the Volterra integral equation (8.28)

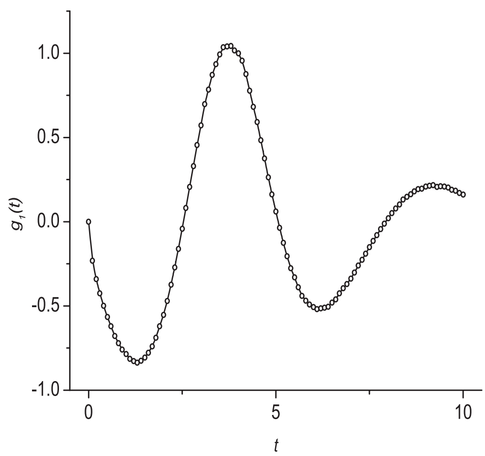


Figure 8.4. *The Neumann data $g_1(t)$ for the case* (8.39).

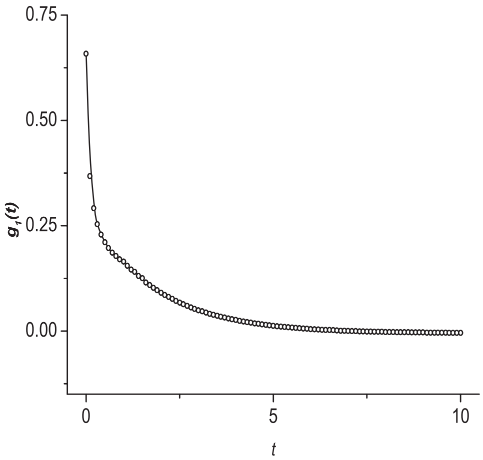


Figure 8.5. *The Neumann data $g_1(t)$ for the case* (8.40).

was solved on a grid of 100 points in the interval $t \in [0, 10]$. In the first case, shown in Figure 8.4, the behavior of $g_1(t)$ is characterized by the nature of the periodic boundary function $g_0(t)$ ($g_1(t)$ oscillates with amplitude that decreases in time). In the second case shown in Figure 8.5, the behavior of $g_1(t)$ is strongly determined by the exponential term in $q_0(x)$ ($g_1(t)$ decays rapidly toward zero).

Part III

Novel Integral Representations for Linear Boundary Value Problems

Novel representations of the solution of initial-boundary value problems for evolution PDEs on the half-line and the finite interval were presented in Part I. These solutions were derived through the analysis of two basic equations formulated in the complex k-plane, namely, (a) the global relation, and (b) the integral representation of the solution.

It turns out that it is straightforward to derive the global relation for a large class of PDEs. Indeed, suppose that the function $q(x, y)$ satisfies a linear PDE in an open piecewise smooth domain $\Omega \in \mathbb{R}^2$. Let $\tilde{q}$ satisfy the formal adjoint of this PDE and *assume* that there exists a one-parameter family of solutions for $\tilde{q}$, i.e., $\tilde{q} = \tilde{q}(x, y; k)$, $k \in \mathbb{C}$. Then, it is always possible to rewrite the given PDE as a one-parameter family of divergence forms, and this immediately yields a global relation.

The above one-parameter family of divergence forms also immediately yields a Lax pair formulation for the given PDE. The *simultaneous spectral analysis* of this Lax pair yields an integral representation of the solution. The level of difficulty of the spectral analysis depends on the nature of the domain. For a polygonal domain the spectral analysis is simple and it is actually conceptually similar to that of Examples 6.1 and 6.2 of Chapter 6. On the other hand, the spectral analysis for an arbitrary domain is rather complicated and is similar to that performed in Chapter 8.

In what follows we will first derive the global relation and the Lax pair formulation of several PDEs. Then, we will rederive the integral representations for evolution PDEs on the half-line and the finite interval (i.e., (1.16) and (2.6), respectively) using the spectral analysis of an appropriate Lax pair. Finally, we will derive integral representations for the basic elliptic PDEs in a polygonal domain.

Chapter 9

Divergence Formulation, the Global Relation, and Lax Pairs

Example 9.1 (the heat equation). Let $q(x,t)$ satisfy the heat equation

$$q_t = q_{xx}. \tag{9.1}$$

The formal adjoint $\tilde{q}(x,t)$ satisfies the equation

$$-\tilde{q}_t = \tilde{q}_{xx}. \tag{9.2}$$

Multiplying (9.1) and (9.2) by $\tilde{q}$ and q, respectively, and then subtracting the resulting equations we find

$$(q\tilde{q})_t - (\tilde{q}q_x - q\tilde{q}_x)_x = 0. \tag{9.3}$$

Suppose that the heat equation is valid in an open, piecewise smooth domain $\Omega \in \mathbb{R}^2$ of the (x,t)-plane and let $\tilde{\Omega}$ be a subdomain of Ω. Then applying Green's theorem to (9.3) in the subdomain $\tilde{\Omega}$ we find

$$\int_{\partial\tilde{\Omega}} \left[(q\tilde{q})d\xi + (\tilde{q}q_\xi - q\tilde{q}_\xi)ds\right] = 0, \tag{9.4}$$

where $\partial\tilde{\Omega}$ denotes the boundary of $\tilde{\Omega}$ and $q, \tilde{q}$ are functions of s and ξ.

A one-parameter family of solutions of (9.2) is given by

$$\tilde{q}(x,t;k) = e^{-ikx+k^2t}, \quad k \in \mathbb{C}.$$

Then (9.3) and (9.4) become, respectively,

$$\left(e^{-ikx+k^2t}q\right)_t - \left[e^{-ikx+k^2t}(q_x + ikq)\right]_x = 0, \quad k \in \mathbb{C}, \tag{9.5}$$

and

$$\int_{\partial\tilde{\Omega}} e^{-ik\xi+k^2s}\left[qd\xi + (q_\xi + ikq)ds\right] = 0, \quad k \in \mathbb{C}. \tag{9.6}$$

In the case that the heat equation is formulated on the half-line, taking

$$\tilde{\Omega} = \{0 < \xi < \infty, 0 < s < t\},$$

(9.6) becomes the global relation (12).

Equation (9.5), in addition to implying the global relation, also immediately yields a Lax pair formulation. Indeed, if the heat equation is valid in Ω, then (9.5) implies the existence of a function M such that

$$M_x = e^{-ikx+k^2t}q, \quad M_t = e^{-ikx+k^2t}(q_x + ikq), \quad (x,t) \in \Omega, \quad k \in \mathbb{C}. \tag{9.7}$$

Letting $M(x,t,k) = \mu(x,t,k)\exp[-ikx + k^2t]$, equations (9.7) yield the following Lax pair for the heat equation:

$$\mu_x - ik\mu = q, \quad \mu_t + k^2\mu = q_x + ikq, \quad (x,t) \in \Omega, \quad k \in \mathbb{C}. \tag{9.8}$$

The above equations can be rewritten in the language of differential forms: Equation (9.5) implies that the following differential form is closed:

$$W(x,t,k) \doteqdot e^{-ikx+k^2t}[qdx + (q_x + ikq)dt]. \tag{9.9}$$

This of course can be verified directly:

$$\begin{aligned} dW &= dx W_x + dt W_t \\ &= \left[e^{-ikx+k^2t}(q_x + ikq)_x\right] dx \wedge dt + \left(e^{-ikx+k^2t}q\right)_t dt \wedge dx \\ &= e^{-ikx+k^2t}(q_{xx} - q_t) dx \wedge dt, \end{aligned}$$

where we have used the fact that the operator $\wedge$ is skew symmetric. Hence, the differential form W is closed if and only if q satisfies the heat equation.

The Poincaré lemma,

$$\int_{\partial\Omega} W = \int\int_{\Omega} dW, \tag{9.10}$$

immediately implies (9.6). Furthermore, if Ω is an open, piecewise smooth domain, the closed differential form W is also exact, i.e., there exists a 0-form $\mu \exp[-ikx + k^2t]$ such that

$$d\left[\mu e^{-ikx+k^2t}\right] = W, \quad (x,t) \in \Omega, \quad k \in \mathbb{C}, \tag{9.11}$$

which, using the definition of W (equation (9.9)), yields the Lax pair (9.8).

It turns out that for the derivation of the integral representation of the solution, the formulation (9.11) in terms of a differential form is more convenient than the Lax pair formulation (9.8). Indeed, instead of performing the simultaneous spectral analysis of (9.8), it is more convenient to perform the spectral analysis of the differential form (9.11).

Remark 9.1. It was shown above that the heat equation is equivalent to the statement that the differential form W is closed. Rewriting PDEs in terms of differential forms has a long and illustrious history, in particular regarding the geometric and group-theoretic properties of PDEs. The novelty of the above formulation is the occurrence of the complex parameter k. Indeed, for the employment of the global relation and for the spectral analysis, it is crucial that a given PDE is rewritten as a *one-parameter family of closed differential forms.*

Example 9.2 (a general evolution PDE). It is straightforward to verify that the following differential form W is closed if and only if $q(x,t)$ satisfies the PDE (1.1):

$$W(x,t,k) = e^{-ikx+w(k)t}\left[qdx + \sum_{j=0}^{n-1} c_j(k)\partial_x^j q dt\right], \quad k \in \mathbb{C}, \tag{9.12}$$

where the constants $\{c_j(k)\}_{j=0}^{n-1}$ are defined by (1.15). Hence, if the PDE (1.1) is valid in the domain Ω, the associated global relation is

$$\int_{\partial\Omega} W(x,t,k) = 0, \quad k \in \mathbb{C}. \tag{9.13}$$

In the particular case that Ω is defined by $\{0 < \xi < \infty, 0 < s < T\}$, (9.13) becomes (1.18). We note that in this case k is restricted by $\operatorname{Im} k \leq 0$, in order for $\hat{q}(k,t)$ and $\hat{q}_0(k)$ to make sense. Similarly, if Ω is defined by $\{0 < \xi < L, 0 < s < t\}$, (9.13) becomes (2.8).

It will be shown in Chapter 10 that the integral representations (1.16) and (2.6) can be rederived by performing the spectral analysis of the differential form

$$d\left[e^{-ikx+w(k)t}\mu(x,t,k)\right] = W(x,t,k), \quad (x,t) \in \Omega, \quad k \in \mathbb{C}. \tag{9.14}$$

Example 9.3 (the modified Helmholtz equation). By employing the formal adjoint, it was shown in the introduction that the modified Helmholtz equation, i.e., the equation

$$q_{xx} + q_{yy} - 4\beta^2 q = 0, \quad \beta > 0, \tag{9.15}$$

can be written in the divergence form (55) of the introduction. This implies that $q(x,y)$ satisfies the modified Helmholtz equation if and only if the following differential form W is closed:

$$W(x,y,k) = e^{k_1 x + k_2 y}\left[(k_2 q - q_y)dx + (q_x - k_1 q)dy\right], \quad k \in \mathbb{C}, \tag{9.16}$$

where $k_1^2 + k_2^2 = 4\beta^2$. In the introduction, the relevant formulae were subsequently rewritten in terms of the variables $(z, \bar{z}, k)$. In what follows we work directly with the latter variables. Furthermore, we use the fact that we can add to W any closed form without affecting the final result (also we can multiply the given PDE by an arbitrary constant). Sometimes this arbitrariness can be helpful in the associated spectral theory.

Letting

$$z = x + iy, \quad \bar{z} = x - iy, \tag{9.17}$$

it follows that

$$\partial_x = \partial_z + \partial_{\bar{z}}, \quad \partial_y = i\left(\partial_z - \partial_{\bar{z}}\right). \tag{9.18}$$

Hence, the modified Helmholtz equation can be rewritten in the form

$$q_{z\bar{z}} - \beta^2 q = 0. \tag{9.19}$$

The formal adjoint $\tilde{q}$ satisfies the same equation as q. A one-parameter family of solutions of (9.19) is given by

$$e^{-i\beta\left(kz - \frac{\bar{z}}{k}\right)}, \quad k \in \mathbb{C}.$$

The function $q(z,\bar{z})$ satisfies the modified Helmholtz equation if and only if the following differential form W is closed for all $k \in \mathbb{C}$:

$$W(z,\bar{z},k) = e^{-i\beta\left(kz-\frac{\bar{z}}{k}\right)}\left[(q_z - ik\gamma\beta q)dz + \left(\gamma q_{\bar{z}} - \frac{\beta}{ik}q\right)d\bar{z}\right], \tag{9.20}$$

where γ is an arbitrary parameter different from 1. Indeed,

$$dW = (1-\gamma)e^{-i\beta\left(kz-\frac{\bar{z}}{k}\right)}\left(q_{z\bar{z}} - \beta^2 q\right)d\bar{z}\wedge dz.$$

If z is on the boundary of a polygon and this boundary is parametrized by s, then

$$q_z dz = \frac{1}{2}\left(\dot{q} + iq_n\right)ds, \tag{9.21}$$

where $\dot{q}$ denotes the derivative of q along the boundary of the polygon and q_n denotes the derivative normal to the boundary. In this case it is convenient to have a differential form which does *not* involve $\dot{q}$. This can be achieved by taking $\gamma = -1$, and thus

$$W = e^{-i\beta\left(kz-\frac{\bar{z}}{k}\right)}\left[(q_z + ik\beta q)dz - \left(q_{\bar{z}} + \frac{\beta}{ik}q\right)d\bar{z}\right], \quad k \in \mathbb{C}. \tag{9.22}$$

Hence, if $\partial\Omega$ denotes the boundary of a convex polygon Ω, the associated global relation is

$$\int_{\partial\Omega} e^{-i\beta\left(kz-\frac{\bar{z}}{k}\right)}\left[iq_n + i\beta\left(k\frac{dz}{ds} + \frac{1}{k}\frac{d\bar{z}}{ds}\right)q\right]ds = 0, \quad k \in \mathbb{C}, \tag{9.23}$$

which coincides with the first of equations (52) of the introduction. It will be shown in Chapter 10 that the spectral analysis of the equation

$$d\left[e^{-i\beta\left(kz-\frac{\bar{z}}{k}\right)}\mu(z,\bar{z},k)\right] = W(z,\bar{z},k), \quad z \in \Omega, \tag{9.24}$$

where $k \in \mathbb{C}$ and W is defined by (9.22), yields the integral representation (51) presented in the introduction.

It turns out that second order elliptic PDEs require *two* global relations. If q is real, then a second global relation can be obtained by taking the Schwarz conjugate of the global relation (9.23), i.e., by taking the complex conjugate of (9.22) and then replacing k by $\bar{k}$. This yields

$$\int_{\partial\Omega} e^{i\beta\left(k\bar{z}-\frac{z}{k}\right)}\left[iq_n + i\beta\left(k\frac{d\bar{z}}{ds} + \frac{1}{k}\frac{dz}{ds}\right)q\right]ds = 0, \tag{9.25}$$

where $k \in \mathbb{C}$, which coincides with the second of equations (52) of the introduction. Actually, (9.25) is valid *without* the assumption that q is real. Indeed, replacing k by $1/k$ in the RHS of (9.22), it follows that if $W(z,\bar{z},k)$ is closed, then $W(z,\bar{z},1/k)$ is also closed; (9.25) follows from the latter differential form.

Example 9.4 (the Helmholtz equation). Replacing β by $i\beta$, (9.15) and (9.19) become the Helmholtz equation in Cartesian and complex coordinates,

$$q_{xx} + q_{yy} + 4\beta^2 q = 0, \quad \beta > 0, \tag{9.26}$$

and

$$q_{z\bar{z}} + \beta^2 q = 0, \quad \beta > 0. \tag{9.27}$$

Novel integral representations for the Helmholtz equation can be obtained by performing the spectral analysis of the differential form

$$d\left[e^{-i\beta(kz+\frac{\bar{z}}{k})}\mu(z,\bar{z},k)\right] = W(z,\bar{z},k), \quad z \in \Omega, \tag{9.28}$$

where $k \in \mathbb{C}$ and W is defined by

$$W = e^{-i\beta(kz+\frac{\bar{z}}{k})}\left[(q_z + ik\beta q)\,dz - \left(q_{\bar{z}} + \frac{i\beta}{k}q\right)d\bar{z}\right], \quad k \in \mathbb{C}. \tag{9.29}$$

The associated global relations are

$$\int_{\partial\Omega} e^{-i\beta(kz+\frac{\bar{z}}{k})}\left[q_n + \beta\left(k\frac{dz}{ds} - \frac{1}{k}\frac{d\bar{z}}{ds}\right)q\right]ds = 0, \quad k \in \mathbb{C}, \tag{9.30}$$

and

$$\int_{\partial\Omega} e^{i\beta(kz+\frac{\bar{z}}{k})}\left[q_n + \beta\left(k\frac{d\bar{z}}{ds} - \frac{1}{k}\frac{dz}{ds}\right)q\right]ds = 0, \quad k \in \mathbb{C}. \tag{9.31}$$

Equations (9.28)–(9.31) follow from (9.22)–(9.25) by replacing β with $i\beta$ and then (for convenience) replacing k with $-ik$.

Example 9.5 (the Helmholtz equation in cylindrical coordinates). Let $q(\rho,\theta)$ satisfy the Helmholtz equation in cylindrical coordinates, i.e.,

$$\rho q_{\rho\rho} + \frac{1}{\rho}q_{\theta\theta} + q_\rho - 4\beta^2\rho q = 0. \tag{9.32}$$

The formal adjoint satisfies the equation

$$(\rho\tilde{q})_{\rho\rho} + \frac{1}{\rho}\tilde{q}_{\theta\theta} - \tilde{q}_\rho - 4\beta^2\rho q = 0. \tag{9.33}$$

Simplifying this equation, it follows that $\tilde{q}$ satisfies the same equation as q, i.e., (9.32). The equations for q and $\tilde{q}$ imply

$$\left(\frac{q}{\rho}\tilde{q}_\theta - \frac{\tilde{q}}{\rho}q_\theta\right)_\theta - (\rho\tilde{q}q_\rho - \rho q\tilde{q}_\rho)_\rho = 0. \tag{9.34}$$

Letting $\tilde{q}(\rho,\theta;k) = \Theta(\theta;k)R(\rho;k)$, it follows that $\Theta = \exp[\pm ik\theta]$ and that R satisfies the ODE

$$\rho^2\ddot{R} + \rho\dot{R} - (4\beta^2\rho^2 + k^2)R = 0. \tag{9.35}$$

Hence, the two global relations associated with the Helmholtz equation in cylindrical coordinates are the equations

$$\int_{\partial\Omega} e^{\pm ik\theta}\left[R(-q_\theta \pm ikq)\frac{d\rho}{\rho} + \rho(Rq_\rho - \dot{R}q)d\theta\right] = 0, \quad k \in \mathbb{C}, \tag{9.36$^\pm$}$$

where R is any solution of the Bessel equation (9.35).

Example 9.6 (the Laplace equation). Using complex coordinates, the Laplace equation takes the simple form

$$q_{z\bar{z}} = 0.$$

Hence, since $(q_z)_{\bar{z}} = 0$, it follows that q is harmonic if and only if q_z is an analytic function of z. This implies that it is easier to obtain an integral representation for q_z, i.e., an integral representation for an analytic function, than an integral representation for q. In this respect we note that the function q satisfies the Laplace equation if and only if the following differential form is closed:

$$W(z,k) = e^{-ikz} q_z dz, \quad k \in \mathbb{C}. \tag{9.37}$$

It will be shown in Chapter 11 that the spectral analysis of the differential form

$$d\left[e^{-ikz}\mu(z,k)\right] = e^{-ikz} q_z dz, \quad k \in \mathbb{C}, \tag{9.38}$$

yields a novel integral representation for q_z in the interior of the convex polygon Ω. The associated global relations are

$$\int_{\partial\Omega} e^{-ikz} q_z dz = 0, \quad \int_{\partial\Omega} e^{ik\bar{z}} q_{\bar{z}} d\bar{z} = 0, \quad k \in \mathbb{C}. \tag{9.39}$$

If q is real, then the second of equations (9.39) follows from the Schwarz conjugation of the first of equations (9.39). If q is complex, the second of equations (9.39) is a consequence of the following differential form which is also closed:

$$\tilde{W}(\bar{z},k) = e^{ik\bar{z}} q_{\bar{z}} d\bar{z}, \quad k \in \mathbb{C}. \tag{9.40}$$

An alternative differential form associated with the Laplace equation is given by

$$W(z,\bar{z},k) = e^{-ikz}[(ikq + q_z)dz - q_{\bar{z}} d\bar{z}].$$

This differential form has the advantage that it involves q, and thus it is more convenient for the solution of the Dirichlet problem; see [25], [40]. However, in this book, for simplicity we will only use the differential form defined by (9.37).

Example 9.7 (separable PDEs with variable coefficients). Let $q(x,t)$ satisfy the time-dependent Schrödinger equation with a time-independent potential,

$$iq_t + q_{xx} + u(x)q = 0. \tag{9.41}$$

The formal adjoint satisfies

$$-i\tilde{q}_t + \tilde{q}_{xx} + u(x)\tilde{q} = 0. \tag{9.42}$$

Hence,

$$(q\tilde{q})_t - i(q_x\tilde{q} - q\tilde{q}_x)_x = 0.$$

Thus, $q(x,t)$ satisfies (9.41) if and only if the following differential form is closed:

$$W = q\tilde{q}dx + i(q_x\tilde{q} - q\tilde{q}_x)dt, \tag{9.43}$$

where $\tilde{q}$ is any solution of (9.42). Using separation of variables it follows that

$$\tilde{q} = e^{ik^2 t}\Phi(x,k), \qquad \Phi_{xx} + (u(x) + k^2)\Phi = 0. \tag{9.44}$$

It is shown in [35] that the spectral analysis of the differential form

$$d\left[\mu(x,k)e^{ik^2 t}\right] = e^{ik^2 t}\left\{\Phi q dt + i\left(\Phi q_x - \Phi_x q\right)dx\right\} \tag{9.45}$$

yields an integral representation for the solution of (9.41) on the half-line.

Similarly, let $q(x, y)$ satisfy

$$q_{xx} + q_{yy} + u(x)q = 0. \tag{9.46}$$

Then

$$(q_x\tilde{q} - q\tilde{q}_x)_x - (q\tilde{q}_y - q_y\tilde{q})_y = 0,$$

where $\tilde{q}$ is a solution of (9.46). Thus $q(x, y)$ satisfies (9.46) if and only if the following differential form is closed:

$$W = (q_x\tilde{q} - q\tilde{q}_x)dy + (q\tilde{q}_y - q_y\tilde{q})dx.$$

Using separation of variables it follows that $\tilde{q} = \exp[ky]\Phi(x,k)$, where $\Phi(x,k)$ satisfies the time-independent Schrödinger equation, i.e., the second of equations (9.44). It is shown in [35] that the spectral analysis of the differential form

$$d\left[\mu(x,k)e^{ky}\right] = e^{ky}\left[(\Phi q_x - \Phi_x q)dy + \Phi(kq - q_y)dx\right]$$

yields an integral representation for the solution of (9.46) in the quarter plane.

The equation

$$q_t + q_{xxx} + u(x)q = 0 \tag{9.47}$$

on the half-line is analyzed in [38].

Chapter 10

Rederivation of the Integral Representations on the Half-Line and the Finite Interval

Let the differential form $W(x,t,k)$ be defined by (9.12) and suppose that $q(x,t)$ satisfies (1.1) in the domain

$$\Omega = \{0 < x < \infty, \ \ 0 < t < T\}. \tag{10.1}$$

We will perform the spectral analysis of (9.14) by following steps similar to those used in Chapter 6.

In order to solve the direct problem, we integrate (9.14) by employing the fundamental theorem of calculus. This implies that for all $(x,t) \in \Omega$,

$$\mu_j(x,t,k) = \int_{(x_j,t_j)}^{(x,t)} e^{ik(x-\xi)-w(k)(t-s)} \left[q(\xi,s)d\xi + \sum_{j=0}^{n-1} c_j(k)\partial_\xi^j q(\xi,s)ds \right]. \tag{10.2}$$

The notation μ_j indicates that the function μ_j depends only on the point (x_j,t_j) and *not* on the path of integration. We must now choose the points (x_j,t_j) in such a way that we can define a sectionally analytic function $\mu(x,t,k)$. It was shown in [22] that if Ω is a polygonal domain, then the proper choice of (x_j,t_j) is the set of the *corners* of this polygon. In our case the domain Ω has three corners, and thus we let (see Figure 10.1)

$$(x_1,t_1) = (0,T), \quad (x_2,t_2) = (0,0), \quad (x_3,t_3) = (\infty,t).$$

The functions $\{\mu_j\}_{j=1}^3$ for all $(x,t) \in \Omega$ are bounded and analytic in the domains of the complex k-plane indicated below,

$$\mu = \begin{cases} \mu_1, & D^+, \\ \mu_2, & E^+, \\ \mu_3, & \operatorname{Im} k \le 0, \end{cases} \tag{10.3}$$

where

$$D^+ = \{k \in \mathbb{C}, \quad \operatorname{Im} k > 0, \quad \operatorname{Re} w(k) < 0\}, \tag{10.4a}$$

$$E^+ = \{k \in \mathbb{C}, \quad \operatorname{Im} k > 0, \quad \operatorname{Re} w(k) > 0\}. \tag{10.4b}$$

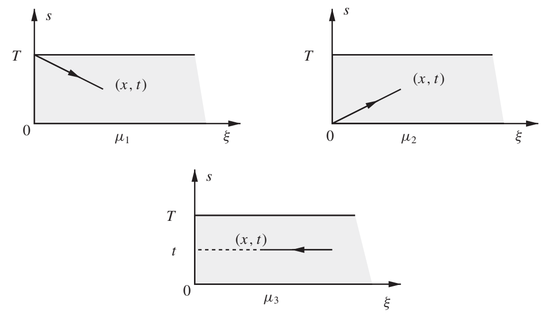


Figure 10.1. *The contours of integration for* $\{\mu_j\}_{j=1}^3$.

Indeed, for the function μ_1, if (ξ, s) is on the path of integration, then $x - \xi \geq 0$ and $t - s \leq 0$. Hence, the exponential occurring in the RHS of (10.2) is bounded as $k \to \infty$ in D^+. Similarly, for the function μ_2, $x - \xi \geq 0$ and $t - s \geq 0$, and thus the exponential occurring in the RHS of (10.2) is bounded as $k \to \infty$ in E^+. For the function μ_3, $s = t$, and thus $ds = 0$, hence (10.2) yields

$$\mu_3(x, t, k) = \int_\infty^x e^{ik(x-\xi)} q(\xi, t) d\xi, \quad \operatorname{Im} k \leq 0, \tag{10.5}$$

and since $x - \xi \leq 0$, this function is well defined for $\operatorname{Im} k \leq 0$.

If T is finite, then the functions μ_1 and μ_2 are entire functions of k which are bounded as $k \to \infty$ in D^+ and E^+, respectively. If $T = \infty$, then μ_1 and μ_2 are defined only if k is restricted in D^+ and E^+, respectively. In the domain where μ is bounded as $k \to \infty$, integration by parts implies that

$$\mu = O\left(\frac{1}{k}\right), \quad k \to \infty. \tag{10.6}$$

In order to solve the inverse problem we must compute the "jumps" of the functions μ_j. The difference $\Delta\mu$ of any two solutions satisfies the equation

$$d\left[(\Delta\mu) e^{-ikx + w(k)t}\right] = 0,$$

and hence the difference of any two solutions is proportional to $\exp[ikx - w(k)t]$. For example,

$$\mu_3(x, t, k) - \mu_2(x, t, k) = e^{ikx - w(k)t} \rho_{23}(k), \quad k \in \{E^+ \cap \mathbb{R}\}, \tag{10.7}$$

where $\rho_{23}(k)$ depends on k (and not on x, t). The function $\rho_{23}(k)$ can be determined by evaluating (10.7) at any point in the (x, t)-plane. For example, letting $x = t = 0$ and using $\mu_2(0, 0, k) = 0$, $\mu_3(0, 0, k) = -\hat{q}_0(k)$, we find $\rho_{23} = -\hat{q}_0(k)$, where $\hat{q}_0(k)$ denotes the Fourier transform of $q_0(x) = q(x, 0)$; see (1.6). Similarly,

$$\mu_1(x, t, k) - \mu_3(x, t, k) = e^{ikx - w(k)t} \rho_{13}(k), \quad k \in \{D^+ \cap \mathbb{R}\}. \tag{10.8}$$

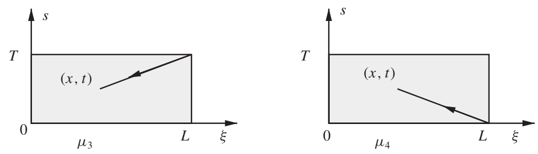


Figure 10.2. *The contour of integration for μ_3 and μ_4.*

Evaluating this equation at $x = t = 0$ and using $\mu_1(0,0,k) = -\tilde{g}(k)$ we find $\rho_{13} = -\tilde{g}(k) + \hat{q}_0(k)$, where

$$\tilde{g}(k) = \int_0^T e^{w(k)s} \sum_{j=0}^{n-1} c_j(k)\partial_\xi^j q(0,s)ds, \quad k \in \mathbb{C}. \tag{10.9}$$

For the evaluation of $\mu_1(0,0,k)$ we have used the fact that if $(x,t) = (0,0)$, then the contour of integration is along the axis $\xi = 0$, and hence $d\xi = 0$. Finally,

$$\mu_1(x,t,k) - \mu_2(x,t,k) = -e^{ikx-w(k)t}\tilde{g}(k), \quad k \in \left\{E^+ \cap D^+/\mathbb{R}\right\}. \tag{10.10}$$

The estimate (10.6) and the "jump" conditions (10.7), (10.8), (10.10) define a Riemann–Hilbert (RH) problem whose unique solution is

$$\mu(x,t,k) = \frac{1}{2i\pi}\left\{\int_{-\infty}^{\infty} e^{ilx-w(l)t}\hat{q}_0(l)\frac{dl}{l-k} - \int_{\partial D^+} e^{ilx-w(l)t}\tilde{g}(l)\frac{dl}{l-k}\right\},$$

where

$$(x,t) \in \Omega, \quad k \in \mathbb{C}\backslash\{\mathbb{R} \cup \partial D^+\}. \tag{10.11}$$

Substituting this expression in the x-part of the associated Lax pair we find (1.16).

In the case of the finite interval, the relevant domain is

$$\Omega = \{0 < x < L, \quad 0 < t < T\}. \tag{10.12}$$

In this case, in addition to μ_1 and μ_2, we also define μ_3 and μ_4 as shown in Figure 10.2. Then

$$\mu = \begin{cases} \mu_1, & D^+, \\ \mu_2, & E^+, \\ \mu_3, & D^-, \\ \mu_4, & E^-, \end{cases} \tag{10.13}$$

where

$$D^- = \{k \in \mathbb{C}, \quad \operatorname{Im} k < 0, \quad \operatorname{Re} w(k) < 0\},$$
$$E^- = \{k \in \mathbb{C}, \quad \operatorname{Im} k < 0, \quad \operatorname{Re} w(k) > 0\}. \tag{10.14}$$

Computing the "jumps" of the functions $\{\mu_j\}_1^4$ and proceeding as in the case of the half-line, we find the integral representation (2.6).

Remark 10.1. The above derivation, in contrast to the derivation presented in Part I, has the advantage that it can be generalized to integrable nonlinear evolution PDEs; see Part V. Furthermore, it provides a spectral interpretation of the novel integral representations (1.16) and (2.6). As mentioned in the introduction, these representations provide the concrete realization of the Euler–Ehrenpreis–Palamodov representations. We recall that the classical representations in terms of the Fourier transform are associated with the spectral theory of certain ordinary differential operators. Is there a spectral theory associated with the Euler–Ehrenpreis–Palamodov representations? The answer is affirmative at least in the case of two dimensions: It is the spectral analysis of an appropriate differential form!

Chapter 11
The Basic Elliptic PDEs in a Polygonal Domain

In this chapter we will derive integral representations for the basic elliptic PDEs in polygonal domains. These representations involve both the Dirichlet as well as the Neumann boundary values. For certain simple domains, by using the global relations it is possible to eliminate either the Dirichlet or the Neumann boundary values. This will be illustrated in Part IV.

For the implementation of these formulae it is useful to recall the following identities.

(a) If

$$z = x + iy, \quad \bar{z} = x - iy, \quad (x, y) \in \mathbb{R}^2,$$

then

$$\partial_z = \frac{1}{2}(\partial_x - i\partial_y), \quad \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y). \tag{11.1}$$

(b) If a side of a polygon is parametrical by s, then

$$q_z dz = \frac{1}{2}(\dot{q} + iq_n)ds, \;\; q_{\bar{z}} d\bar{z} = \frac{1}{2}(\dot{q} - iq_n)ds, \tag{11.2}$$

where $\dot{q}$ is the derivative along the side, i.e., $\dot{q} = dq(z(s))/ds$, and q_n is the derivative normal to the side in the outward direction.

11.1 The Laplace Equation in a Convex Polygon

By performing the spectral analysis of the differential form (9.38) we will derive the following result [91].

Proposition 11.1. Let Ω be the interior of a convex bounded polygon in the complex z-plane, with corners $z_1, \ldots, z_n, z_{n+1} = z_1$; see Figure 6 of the introduction. Assume that there exists a solution $q(z, \bar{z})$ of the Laplace equation valid in the interior of Ω and assume that this solution has sufficient smoothness all the way to the boundary of the polygon. Then q_z can be expressed in the form

$$\frac{\partial q}{\partial z} = \frac{1}{2\pi} \sum_{j=1}^{n} \int_{l_j} e^{ikz} \hat{q}_j(k) dk, \quad z \in \Omega, \tag{11.3}$$

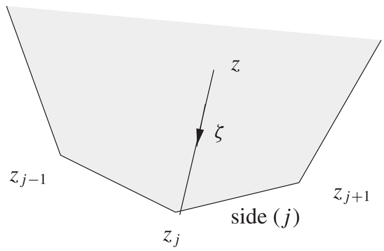


Figure 11.1.

where the functions $\{\hat{q}_j(k)\}_1^n$ are defined by

$$\hat{q}_j(k) = \int_{z_j}^{z_{j+1}} e^{-ikz} q_z dz, \quad k \in \mathbb{C}, \quad j = 1, \dots, n, \tag{11.4}$$

and the rays $\{l_j\}_1^n$ are defined by

$$l_j = \{k \in \mathbb{C}, \ \arg k = -\arg(z_{j+1} - z_j)\}, \quad j = 1, \dots, n, \tag{11.5}$$

and are directed toward infinity.

Furthermore, the following global relations are valid:

$$\sum_{j=1}^{n} \hat{q}_j(k) = 0, \quad \sum_{j=1}^{n} \tilde{q}_j(k) = 0, \quad k \in \mathbb{C}, \tag{11.6}$$

where $\{\tilde{q}_j(k)\}_1^n$ are defined by

$$\tilde{q}_j(k) = \int_{z_j}^{z_{j+1}} e^{ik\bar{z}} q_{\bar{z}} d\bar{z}, \quad k \in \mathbb{C}, \quad j = 1, \dots, n. \tag{11.7}$$

Proof. We will perform the spectral analysis of the differential form (9.38) by following the same steps used in Chapter 10.

Integrating (9.38) we find

$$\mu_j(z,k) = \int_{z_j}^{z} e^{ik(z-\zeta)} q_\zeta d\zeta, \quad z \in \Omega, \quad j = 1, \dots, n. \tag{11.8}$$

The term $\exp[ik(z-\zeta)]$ is bounded as $k \to \infty$ for

$$0 \le \arg k + \arg(z - \zeta) \le \pi. \tag{11.9}$$

If z is inside the polygon and ζ is on a curve from z to z_j (see Figure 11.1), then

$$\arg(z_{j+1} - z_j) \le \arg(z - \zeta) \le \arg(z_{j-1} - z_j), \quad j = 1, \dots, n.$$

Hence, the inequalities (11.9) are satisfied, provided that

$$-\arg\left(z_{j+1} - z_j\right) \le \arg k \le \pi - \arg\left(z_{j-1} - z_j\right).$$

Hence, the function μ_j is an entire function of k which is bounded as $k \to \infty$ in the sector Σ_j defined by

$$\Sigma_j = \left\{k \in \mathbb{C}, \ \arg k \in \left[-\arg\left(z_{j+1} - z_j\right), \pi - \arg\left(z_{j-1} - z_j\right)\right]\right\}, \quad j = 1, \ldots, n. \tag{11.10}$$

The angle of the sector Σ_j, which we denote by Ψ_j, equals

$$\Psi_j = \pi - \arg\left(z_{j-1} - z_j\right) + \arg\left(z_{j+1} - z_j\right) = \pi - \phi_j,$$

where ϕ_j is the angle at the corner z_j. Hence,

$$\sum_{j=1}^{n} \Psi_j = n\pi - \sum_{j=1}^{n} \phi_j = n\pi - \pi(n-2) = 2\pi, \tag{11.11}$$

and thus the sectors $\{\Sigma_j\}_1^n$ precisely cover the complex k-plane. Hence, the function

$$\mu = \mu_j, \quad z \in \Omega, \quad k \in \Sigma_j, \quad j = 1, \ldots, n, \tag{11.12}$$

defines a sectionally analytic function in the complex k-plane.

For the solution of the inverse problem, we note that integration by parts implies that $\mu_j = O(1/k)$ as $k \to \infty$ in Σ_j, i.e.,

$$\mu = O\left(\frac{1}{k}\right), \quad k \to \infty. \tag{11.13}$$

Furthermore, by subtracting (11.8) and the analogous equation for μ_{j+1} we find

$$\mu_j - \mu_{j+1} = e^{ikz}\hat{q}_j(k), \quad z \in \Omega, \quad k \in l_j, \quad j = 1, \ldots, n, \tag{11.14}$$

where $\{\hat{q}_j(k)\}_1^n$ are defined by (11.4) and l_j is the ray of overlap of the sectors Σ_j and Σ_{j+1}. Using the identity

$$\pi - \arg\left(z_j - z_{j+1}\right) = -\arg\left(z_{j+1} - z_j\right) \mod(2\pi), \tag{11.15}$$

it follows that l_j is defined by (11.5). Furthermore, Σ_j is to the left of Σ_{j+1}; see Figure 11.2.

The solution of the Riemann–Hilbert (RH) problem defined by (11.12)–(11.14) is given by

$$\mu = \frac{1}{2i\pi} \sum_{j=1}^{n} \int_{l_j} e^{ilz} \hat{q}_j(l) \frac{dl}{l-k}, \quad z \in \Omega, \quad k \in \mathbb{C} \backslash \{l_j\}_1^n. \tag{11.16}$$

Substituting this expression in (9.38), i.e., in equation

$$\mu_z - ik\mu = q_z,$$

we find (11.3).

Using the definitions of $\{\hat{q}_j\}_1^n$ and $\{\tilde{q}_j\}_1^n$, i.e., (11.4) and (11.7), respectively, equations (9.39) yield the two global relations (11.6). ☐

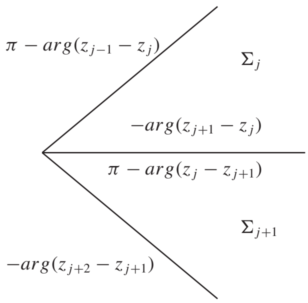


Figure 11.2. *The sectors* Σ_j *and* Σ_{j+1}.

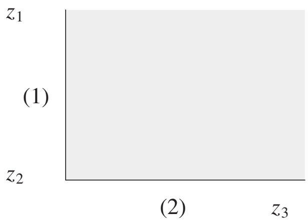


Figure 11.3. *The quarter plane.*

Remark 11.1. It is shown in [91] that a similar result is valid for an unbounded polygon, provided that q_z vanishes at infinity. In this case, if $z_1 = z_n = \infty$, then we define only $n-1$ functions $\hat{q}_j$. Furthermore, $\hat{q}_1$ and $\hat{q}_{n-1}$ are *not* defined for all complex k but only for k in certain sectors of the complex k-plane. The derivation is similar to that of Proposition 11.1, where the proof that $\{\Sigma_j\}_1^{n-1}$ cover the complex k-plane follows from the identity

$$\sum_{j=2}^{n-1} \phi_j = \pi(n-3) + \phi_\infty.$$

Example 11.1 (the quarter plane). Let Ω be the interior of the first quadrant of the complex z-plane; see Figure 11.3.

In this case, $\hat{q}_1$ involves an integral from z_1 to z_2 (where $z = iy$), whereas $\hat{q}_2$ involves an integral from z_2 to z_3 (where $z = x$). Hence, using the first of equations (11.1), (11.4) yields

$$\hat{q}_1(k) = -\frac{i}{2}\int_0^\infty e^{ky}\left[q_x(0,y) - iq_y(0,y)\right]dy, \quad \operatorname{Re} k \le 0, \tag{11.17a}$$

and

$$\hat{q}_2(k) = \frac{1}{2}\int_0^\infty e^{-ikx}\left[q_x(x,0) - iq_y(x,0)\right]dx, \quad \operatorname{Im} k \le 0, \tag{11.17b}$$

where the above restrictions of k ensure that the functions $\hat{q}_1$ and $\hat{q}_2$ make sense.

The first of the global relations (11.6) is defined for k in the domain of the overlap of $\hat{q}_1$ and $\hat{q}_2$, and thus

$$\hat{q}_1(k) + \hat{q}_2(k) = 0, \quad \pi \le \arg k \le \frac{3\pi}{2}. \tag{11.18}$$

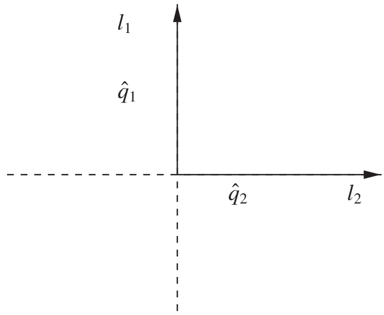


Figure 11.4. *The rays for the quarter plane.*

Figure 11.5. *A semi-infinite strip.*

If q is *not* real, (11.18) must be supplemented with a similar relation involving $\tilde{q}_1$ and $\tilde{q}_2$.

The vector $z_2 - z_1$ makes an angle $-\pi/2$ with the positive x-axis, and hence $-\arg(z_2 - z_1) = \pi/2$, whereas the vector $z_3 - z_2$ is along the positive x-axis, and hence $\arg(z_3 - z_2) = 0$. Thus,

$$q_z = \frac{1}{2\pi}\left[\int_{l_1} e^{ikz}\hat{q}_1(k)dk + \int_{l_2} e^{ikz}\hat{q}_2(k)dk\right], \quad z \in \Omega, \tag{11.19}$$

where the rays l_1 and l_2 are depicted in Figure 11.4.

The analysis of the global relation (11.18) together with the integral representation (11.19) can be used for the solution of a large class of boundary value problems; see [29] and Part IV. In this respect we note that for a well-posed problem, each of $\hat{q}_1$ and $\hat{q}_2$ involves one unknown function, and thus the single equation (11.18) must be solved for two unknown functions.

Example 11.2 (the semi-infinite strip). Let Ω be the interior of the semi-infinite strip with finite corners at the origin and at the point $z_2 = il$, where l is a finite positive constant; see Figure 11.5. In this case $\hat{q}_1$ involves an integral from z_1 to z_2 (where $z = x + il$), $\hat{q}_2$ an integral from z_2 to z_3 (where $z = iy$), and $\hat{q}_3$ an integral from z_3 to z_4 (where $z = x$). Hence,

$$\hat{q}_1(k) = -\frac{e^{kl}}{2}\int_0^\infty e^{-ikx}[q_x(x,l) - iq_y(x,l)]dx, \quad \operatorname{Im} k \le 0, \tag{11.20a}$$

$$\hat{q}_2(k) = -\frac{i}{2}\int_0^l e^{ky}[q_x(0,y) - iq_y(0,y)]dy, \quad k \in \mathbb{C}, \tag{11.20b}$$

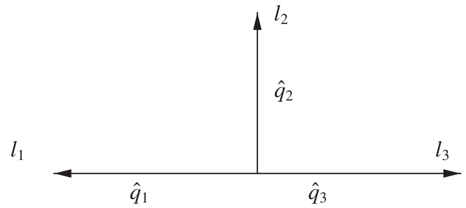


Figure 11.6. *The rays for the semi-infinite strip.*

$$\hat{q}_3(k) = \frac{1}{2}\int_0^{\infty} e^{-ikx}[q_x(x,0) - iq_y(x,0)]dx, \quad \operatorname{Im} k \leq 0. \tag{11.20c}$$

The first of the global relations is

$$\sum_{j=1}^{3} \hat{q}_j(k) = 0, \quad \operatorname{Im} k \leq 0. \tag{11.21}$$

The vector $z_2 - z_1$ makes an angle π with the positive real axis, hence $-\arg(z_2 - z_1) = -\pi$, whereas the vectors $z_3 - z_2$ and $z_4 - z_3$ are similar to those of Example 11.1. Hence,

$$q_z = \frac{1}{2\pi}\sum_{j=1}^{3}\int_{l_j} e^{ikz}\hat{q}_j(k)dk, \quad z \in \Omega, \tag{11.22}$$

where the rays $\{l_j\}_1^3$ are depicted in Figure 11.6.

Remark 11.2. The global relation (11.21) involves *three* unknown functions, whereas (11.18) involves only *two* unknowns. However, (11.21) is valid in a larger domain in the complex k-plane than (11.18). Using this fact it is shown in [29] and in Part IV that (11.21) and (11.22) yield the solution of a large class of boundary value problems for the Laplace equation in a semi-infinite strip.

Example 11.3 (the equilateral triangle). Let Ω be the interior of an equilateral triangle with corners at the following points (see Figure 11.7):

$$z_1 = \frac{l}{\sqrt{3}}e^{-\frac{i\pi}{3}}, \quad z_2 = \bar{z}_1, \quad z_3 = -\frac{l}{\sqrt{3}}.$$

Along the sides (1), (2), (3), z can be parametrized as follows:

$$z(s) = \frac{l}{2\sqrt{3}} + is, \quad z(s) = \left(\frac{l}{2\sqrt{3}} + is\right)\alpha, \quad z(s) = \left(\frac{l}{2\sqrt{3}} + is\right)\bar{\alpha},$$

$$\alpha = e^{\frac{2i\pi}{3}}, \quad s \in \left(-\frac{l}{2}, \frac{l}{2}\right). \tag{11.23}$$

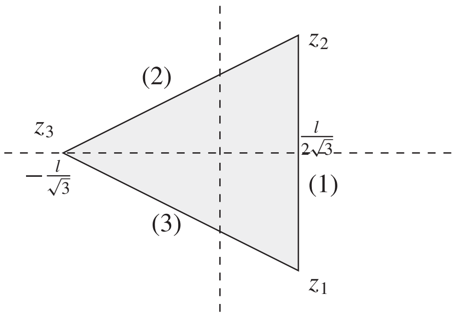


Figure 11.7. *An equilateral triangle.*

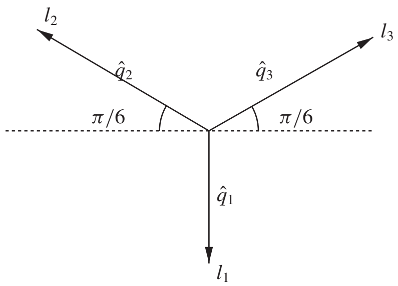


Figure 11.8. *The rays* $\{l_j\}_1^3$ *for the equilateral triangle.*

Using the first of equations (11.2), (11.4) implies

$$\hat{q}_1(k) = \frac{1}{2}e^{-ik\frac{l}{2\sqrt{3}}} \int_{-\frac{l}{2}}^{\frac{l}{2}} e^{ks}\left(\dot{q}^{(1)} + iq_n^{(1)}\right) ds, \quad k \in \mathbb{C}, \tag{11.24a}$$

$$\hat{q}_2(k) = \frac{1}{2}e^{-i\alpha k\frac{l}{2\sqrt{3}}} \int_{-\frac{l}{2}}^{\frac{l}{2}} e^{\alpha ks}\left(\dot{q}^{(2)} + iq_n^{(2)}\right) ds, \quad k \in \mathbb{C}, \tag{11.24b}$$

$$\hat{q}_3(k) = \frac{1}{2}e^{-i\bar{\alpha}k\frac{l}{2\sqrt{3}}} \int_{-\frac{l}{2}}^{\frac{l}{2}} e^{\bar{\alpha}ks}\left(\dot{q}^{(3)} + iq_n^{(3)}\right) ds, \quad k \in \mathbb{C}, \tag{11.24c}$$

where the superscript (1) indicates side (1), etc.

The global relation is

$$\sum_{j=1}^{3} \hat{q}_j(k) = 0, \quad k \in \mathbb{C}. \tag{11.25}$$

The vector $z_2 - z_1$ makes an angle $\pi/2$ with the positive axis, the vector $z_3 - z_2$ an angle $-5\pi/6$, and the vector $z_1 - z_3$ an angle $-\pi/6$. Thus q_z is given by (11.22), where the rays $\{l_j\}_1^3$ are depicted in Figure 11.8.

Remark 11.3. Proposition 11.1 characterizes q_z as opposed to q. For many applications this is sufficient, since in many applications it is precisely the quantity q_z which is physically significant. Also, if q is real, it is straightforward to compute q in terms of q_z: Let $Q(z) = q_z$; then

$$q = \frac{1}{2}(U + \bar{U}), \quad U \doteqdot \int^z Q(z')dz'. \tag{11.26}$$

In the general case

$$q = \frac{1}{2}(U + \tilde{U}), \quad \tilde{U} = \int^{\bar{z}} \tilde{Q}(\bar{z}')d\bar{z}', \tag{11.27}$$

where $\tilde{Q}(\bar{z}) = q_{\bar{z}}$ and $q_{\bar{z}}$ satisfies the following integral representations:

$$q_{\bar{z}} = \frac{1}{2\pi} \sum_{j=1}^{n} \int_{\bar{l}_j} e^{-ik\bar{z}} \tilde{q}_j(k)dk, \quad z \in \Omega, \tag{11.28}$$

where $\{\bar{l}_j\}_{j=1}^n$ are the complex conjugates of $\{l_j\}_{j=1}^n$.

Remark 11.4. Proposition 11.1 implies that if q satisfies Laplace's equation, then necessarily the global relations (11.6) are valid. Actually, these relations are also *sufficient* conditions for the existence of a solution. Indeed, the following result is derived in [92]: Let $\partial\Omega$ denote the boundary of the polygon Ω of Proposition 11.1 and let S_j denote the side from z_j to z_{j+1}. For each $j = 1, \ldots, n$, let $Q^{(j)} \in H^{1/2+\varepsilon}(L_j)$ for $\varepsilon > 0$ with $Q^{(j)}(z_{j+1}) = Q^{(j+1)}(z_{j+1})$ and define $\rho_j(k)$ by the line integral

$$\rho_j(k) = \int_{z_j}^{z_{j+1}} e^{-ikz} Q^{(j)}(z)dz, \quad k \in \mathbb{C}. \tag{11.29}$$

Suppose that the functions $\rho_j(k)$ satisfy the global relation

$$\sum_{j=1}^{n} \rho_j(k) = 0, \quad k \in \mathbb{C}. \tag{11.30}$$

Define $Q(z)$ by

$$Q(z) = \frac{1}{2\pi} \sum_{j=1}^{n} \int_{l_j} e^{ikz} \rho_j(k)dk, \quad z \in \Omega, \tag{11.31}$$

and let q denote the antiderivative of $Q(z)$. Then Q and q are continuous on $\Omega \cup \partial\Omega$ and analytic in Ω, Re (q) satisfies the Laplace equation in Ω, and on each side S_j,

$$q_z = Q = Q^{(j)}, \quad j = 1, \ldots, n. \tag{11.32}$$

11.2 The Modified Helmholtz Equation in a Convex Polygon

By performing the spectral analysis of the differential form (9.24), with W defined by (9.22), we will derive the result of Proposition 1 of the introduction.

Integrating (9.24) we find that for $z \in \Omega$,

$$\mu_j(z,\bar{z},k) = \int_{z_j}^{z} e^{i\beta[k(z-\zeta)-\frac{1}{k}(\bar{z}-\bar{\zeta})]}\left[(q_\zeta + ik\beta q)d\zeta - \left(q_{\bar{\zeta}} + \frac{\beta}{ik}q\right)d\bar{\zeta}\right]. \tag{11.33}$$

This is an entire function of k which is bounded as $k \to \infty$ and $k \to 0$ in the sector Σ_j of the complex k-plane defined by (11.10). Indeed, (11.33) involves the two exponentials

$$e^{i\beta k(z-\zeta)}, \quad e^{-\frac{i\beta}{k}(\bar{z}-\bar{\zeta})} = e^{-\frac{i\beta\bar{k}}{|k|^2}(\bar{z}-\bar{\zeta})}.$$

The real parts of the exponent of these two exponentials have the same sign, and thus the exponentials have identical domains of boundedness as k and $1/k$ tend to infinity.

The differential form (9.24) is equivalent to the following Lax pair:

$$\mu_z - i\beta k = q_z + i\beta k q, \quad \mu_{\bar{z}} + \frac{i\beta}{k}\mu = -\left(q_{\bar{z}} + \frac{\beta}{ik}q\right). \tag{11.34}$$

The first of these equations suggests that

$$\mu = -q + O\left(\frac{1}{k}\right), \quad k \to \infty. \tag{11.35}$$

This can be verified using (11.33) with $k \in \Sigma_j$ and integration by parts. Also subtracting (11.33) and the analogous equation for μ_{j+1} we find

$$\mu_j - \mu_{j+1} = e^{i\beta\left(kz-\frac{\bar{z}}{k}\right)}\hat{q}_j(k), \quad k \in l_j, \tag{11.36}$$

where $\{\hat{q}_j\}_1^n$ are defined by (51b). Using the identities (11.2), which express $q_z dz$ in terms of $\dot{q}$ and q_n, the expression for $\hat{q}_j$ becomes that of (51d).

The solution of the RH problem defined by (11.35) and (11.36) is given for all $k \in \mathbb{C}\setminus\sum_1^n l_j$ by

$$\mu = -q + \frac{1}{2i\pi}\sum_{j=1}^{n}\int_{l_j} e^{i\beta\left(lz-\frac{\bar{z}}{l}\right)}\hat{q}_j(l)\frac{dl}{l-k}, \quad z \in \Omega. \tag{11.37}$$

Substituting this expression in the second of equations (11.34), we find (51a).

Using the definitions of $\{\hat{q}_j\}_1^n$ and $\{\tilde{q}_j\}_1^n$ (i.e., (51b) and (53)) in (9.23) and (9.25), we find the global relations (52). ∎

Example 11.4 (the quarter plane). Let Ω be the interior of the first quadrant of the complex z-plane; see Figure 11.3. Then

$$q(z,\bar{z}) = \frac{1}{4i\pi}\sum_{j=1}^{2}\int_{l_j} e^{i\beta\left(kz-\frac{\bar{z}}{k}\right)}\hat{q}_j(k)\frac{dk}{k}, \quad z \in \Omega, \tag{11.38}$$

where the rays l_1 and l_2 are depicted in Figure 11.4.

Using $z = iy$ and $z = x$ for the sides (1) and (2), respectively, as well as the identities (11.1), (51b) yields

$$\hat{q}_1(k) = -\int_0^\infty e^{\beta(k+\frac{1}{k})y}\left[iq_x(0,y) + \beta\left(\frac{1}{k} - k\right)q(0,y)\right]dy, \quad \mathrm{Re}\, k \le 0, \tag{11.39a}$$

$$\hat{q}_2(k) = \int_0^\infty e^{-i\beta(k-\frac{1}{k})x}\left[-iq_y(x,0) + i\beta\left(\frac{1}{k} + k\right)q(x,0)\right]dx, \quad \mathrm{Im}\, k \le 0. \tag{11.39b}$$

These equations can also be obtained from (51d) using the fact that on the sides (1) and (2) q_n equals q_x and $-q_y$, respectively.

The first of the global relations is (11.18).

Example 11.5 (the semi-infinite strip). Let Ω be the interior of the semi-infinite strip of the complex z-plane described in Example 11.2. Then $q(z,\bar{z})$ is given by (51a) with $n = 3$ and with the rays $\{l_j\}_1^3$ depicted in Figure 11.6.

Using $z = x + il$, $z = iy$, $z = x$ for the sides (1), (2), (3), respectively, as well as the identities (11.1), (51b) yields

$$\hat{q}_1(k) = e^{\beta(k+\frac{1}{k})l}\int_0^\infty e^{-i\beta(k-\frac{1}{k})x}\left[iq_y(x,l) - i\beta\left(k + \frac{1}{k}\right)q(x,l)\right]dx, \tag{11.40a}$$

$$\mathrm{Im}\, k \le 0,$$

$$\hat{q}_2(k) = -\int_0^l e^{\beta(k+\frac{1}{k})y}\left[iq_x(0,y) + \beta\left(\frac{1}{k} - k\right)q(0,y)\right]dy, \quad k \in \mathbb{C}, \tag{11.40b}$$

$$\hat{q}_3(k) = \int_0^\infty e^{-i\beta(k-\frac{1}{k})x}\left[-iq_y(x,0) + i\beta\left(k + \frac{1}{k}\right)q(x,0)\right]dx, \quad \mathrm{Im}\, k \le 0. \tag{11.40c}$$

The first of the global relations is (11.21).

Example 11.6 (the equilateral triangle). Let Ω be the interior of the equilateral triangle described in Example 11.3. Then $q(z,\bar{z})$ is given by (51a) with $n = 3$ and with the rays $\{l_j\}_1^3$ depicted in Figure 11.8.

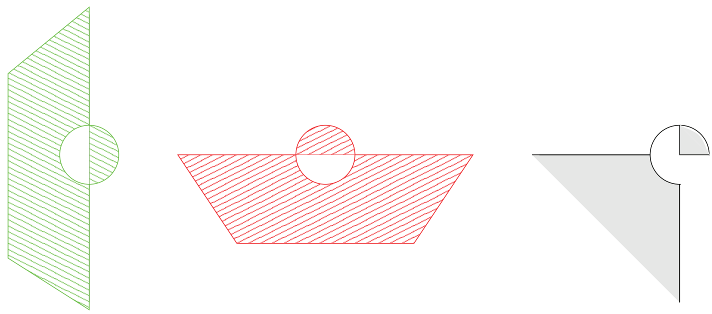
Using the parametrizations (11.23), (51d) implies that for all $k \in \mathbb{C}$,

$$\hat{q}_1(k) = e^{\frac{l}{2\sqrt{3}}(-ik-\frac{1}{ik})}\int_{-\frac{l}{2}}^{\frac{l}{2}} e^{\beta(k+\frac{1}{k})s}\left[iq_n^{(1)} + \beta\left(\frac{1}{k} - k\right)q^{(1)}\right]ds, \tag{11.41a}$$

$$\hat{q}_2(k) = e^{\frac{l}{2\sqrt{3}}(-i\alpha k-\frac{1}{i\alpha k})}\int_{-\frac{l}{2}}^{\frac{l}{2}} e^{\beta(\alpha k+\frac{1}{\alpha k})s}\left[iq_n^{(2)} + \beta\left(\frac{1}{\alpha k} - \alpha k\right)q^{(2)}\right]ds, \tag{11.41b}$$

$$\hat{q}_3(k) = e^{\frac{l}{2\sqrt{3}}(-i\bar{\alpha} k-\frac{1}{i\bar{\alpha} k})}\int_{-\frac{l}{2}}^{\frac{l}{2}} e^{\beta(\bar{\alpha} k+\frac{1}{\bar{\alpha} k})s}\left[iq_n^{(3)} + \beta\left(\frac{1}{\bar{\alpha} k} - \bar{\alpha} k\right)q^{(3)}\right]ds. \tag{11.41c}$$

The first of the global relations is (11.25).



$D_{i\infty}$ (a) D_{∞} (b) D (c)

Figure 11.9. *The domains $D_{i\infty}$, D_{∞}, D.*

11.3 The Helmholtz Equation in the Quarter Plane

The analogues of Proposition 11.1 and of Proposition 1 for the Helmholtz equation are presented in [22]. The relevant spectral analysis is slightly more complicated, and the contour of integration in the complex k-plane, instead of rays, involves unions of rays and circular arcs. Instead of deriving the result presented in [22] we concentrate on the particular case of the quarter plane.

Proposition 11.2. Let Ω be the interior of the first quadrant of the complex z-plane described in Example 11.1. Assume that there exists a solution $q(z, \bar{z})$ of the Helmholtz equation (9.26) (or (9.27)) valid in the interior of Ω, and assume that this solution satisfies the usual radiation condition at infinity and is sufficiently smooth on the boundary of Ω. Then q can be expressed in the form

$$q(z, \bar{z}) = \frac{1}{4\pi i} \sum_{j=1}^{3} \int_{L_j} e^{i\beta\left(kz + \frac{\bar{z}}{k}\right)} \hat{q}_j(k) dk, \quad z \in \Omega, \tag{11.42}$$

where

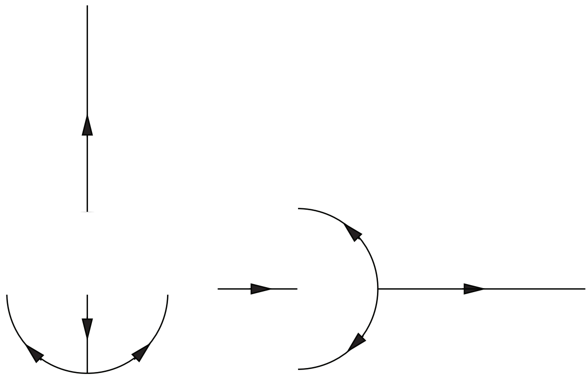
$$\hat{q}_1(k) = -\int_0^\infty e^{\beta\left(k - \frac{1}{k}\right)y} \left[i q_x(0, y) - \beta \left(k + \frac{1}{k} \right) q(0, y) \right] dy, \quad k \in D_{i\infty}, \tag{11.43a}$$

$$\hat{q}_2(k) = -\int_0^\infty e^{-i\beta\left(k + \frac{1}{k}\right)x} \left[-i q_y(x, 0) + i\beta \left(k - \frac{1}{k} \right) q(x, 0) \right] dx, \quad k \in D_{\infty}, \tag{11.43b}$$

the domains $D_{i\infty}$ and D_{∞}, depicted in Figure 11.9, are defined by

$$D_{i\infty} = \Big\{ k \in \mathbb{C}, \{|k| \le 1\} \cap \{\operatorname{Re} k \ge 0\}, \quad \{|k| \ge 1\} \cap \{\operatorname{Re} k \le 0\} \Big\}, \tag{11.44a}$$

$$D_{\infty} = \Big\{ k \in \mathbb{C}, \{|k| \le 1\} \cap \{\operatorname{Im} k \ge 0\}, \quad \{|k| \ge 1\} \cap \{\operatorname{Im} k \le 0\} \Big\}, \tag{11.44b}$$



(a) The curve L_1 (b) The curve L_2

Figure 11.10. *The curves L_1 (green), L_2 (red), L_3 (black).*

and the curves $\{L_j\}_1^2$, depicted in Figure 11.10, are defined by

$$L_1 = \left\{ k \in \mathbb{C}, \quad \{|k| \le 1\} \cap \left\{\arg k = \frac{3\pi}{2}\right\}, \quad \{|k| \ge 1\} \cap \left\{\arg k = \frac{\pi}{2}\right\}, \right.$$
$$\left. \{|k| = 1\} \cap \{-\pi \le \arg k \le 0\} \right\},$$

$$L_2 = \left\{ k \in \mathbb{C}, \quad \{|k| \le 1\} \cap \{\arg k = \pi\}, \quad \{|k| \ge 1\} \cap \{\arg k = 0\}, \right.$$
$$\left. \{|k| = 1\} \cap \left\{-\frac{\pi}{2} \le \arg k \le \frac{\pi}{2}\right\} \right\}.$$

Furthermore, the following global relation is valid:

$$\hat{q}_1(k) = \hat{q}_2(k), \quad k \in D, \tag{11.45}$$

where the domain D, depicted in Figure 11.9, is defined by

$$D = \left\{ k \in \mathbb{C}, \{|k| < 1\} \cap \left\{0 < \arg k \le \frac{\pi}{2}\right\}, \quad \{|k| > 1\} \cap \left\{\pi \le \arg k \le \frac{3\pi}{2}\right\}, \right.$$
$$\left. \{|k| = 1\} \cap \left\{0 < \arg k < \frac{3\pi}{2}\right\} \right\}. \tag{11.46}$$

If q is not real, (11.45) must be supplemented with the equation obtained from (11.45) by replacing i with $-i$ in the definitions of $\hat{q}_1$, $\hat{q}_2$ and by replacing D with $\tilde{D}$, which is the domain obtained from D via Schwarz conjugation.

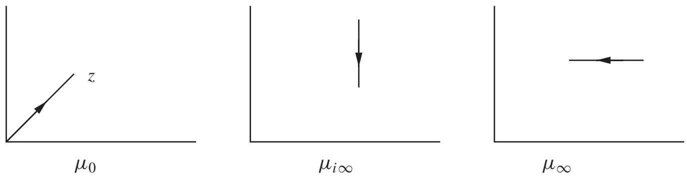


Figure 11.11. *The contours associated with μ_0, $\mu_{i\infty}$, and μ_∞.*

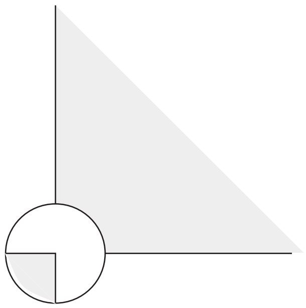


Figure 11.12. *The domain D_0.*

Proof. Integrating (9.28) we find that for all $z \in \Omega$,

$$\mu_j = \int_{z_j}^{z} e^{i\beta\left[k(z-\zeta)+\frac{1}{k}(\bar{z}-\bar{\zeta})\right]} \left[(q_\zeta + ik\beta q)d\zeta - \left(q_{\bar{\zeta}} + \frac{i\beta}{k} q \right) d\bar{\zeta} \right]. \tag{11.47}$$

We choose for z_j the points (0, 0), (x, ∞), and (∞, y) and we denote the resulting solutions by μ_0, $\mu_{i\infty}$, μ_∞, see Figure 11.11. For the latter two functions, $\zeta = x + i\eta$ and $\zeta = \xi + iy$, thus $d\zeta = id\eta$ and $d\zeta = d\xi$, and hence μ_0 is defined by (11.47) with $z_j = 0$, whereas $\mu_{i\infty}$ and μ_∞ are defined as follows:

$$\mu_{i\infty} = \int_{\infty}^{y} e^{\beta\left(\frac{1}{k}-k\right)(y-\eta)} \left[iq_x(x,\eta) - \beta\left(k + \frac{1}{k}\right) q(x,\eta) \right] d\eta, \quad k \in D_{i\infty}, \tag{11.48}$$

$$\mu_\infty = \int_{\infty}^{x} e^{i\beta\left(k+\frac{1}{k}\right)(x-\xi)} \left[-iq_y(\xi,y) + i\beta\left(k - \frac{1}{k}\right) q(\xi,y) \right] d\xi, \quad k \in D_\infty. \tag{11.49}$$

The function μ_0 is an entire function which is bounded as $k \to \infty$ and $k \to 0$ in D_0, depicted in Figure 11.12, defined by

$$D_0 = \left\{ k \in \mathbb{C}, \{|k| \geq 1\} \cap \left\{ 0 \leq \arg k \leq \frac{\pi}{2} \right\}, \right.$$
$$\left. \{|k| \leq 1\} \cap \left\{ \pi \leq \arg k \leq \frac{3\pi}{2} \right\}, \{|k| = 1\} \right\}. \tag{11.50}$$

The functions $\mu_{i\infty}$ and μ_{∞} are defined for k in $D_{i\infty}$ and D_{∞}, respectively. The domains D_0, $D_{i\infty}$, and D_{∞} can be determined as follows: Letting $k = |k|\exp[i\varphi]$, we find

$$\mathrm{Re}\left[ik(z-\zeta) + \frac{i}{k}(\bar{z}-\bar{\zeta})\right] = -\left(|k| - \frac{1}{|k|}\right)[(y-\eta)\cos\varphi + (x-\xi)\sin\varphi]. \quad (11.51)$$

For the function μ_0, $x-\xi \geq 0$ and $y-\eta \geq 0$, and thus the relevant exponential is bounded if and only if

$$\left\{\left(|k| - \frac{1}{|k|}\right) \geq 0\right\} \cap \{\cos\varphi \geq 0 \ \text{ and } \ \sin\varphi \geq 0\}$$

or

$$\left\{\left(|k| - \frac{1}{|k|}\right) \leq 0\right\} \cap \{\cos\varphi \leq 0 \ \text{ and } \ \sin\varphi \leq 0\},$$

which defines D_0. For the function $\mu_{i\infty}$, $x-\xi = 0$ and $y-\eta \leq 0$, which defines $D_{i\infty}$, whereas for μ_{∞}, $x-\xi \leq 0$ and $y-\eta = 0$, which defines D_{∞}. If $|k|^2 = 1$, then the RHS of (11.51) vanishes, thus if k is on the unit circle, all three functions μ_0, $\mu_{i\infty}$, μ_{∞} are bounded.

Using the radiation condition at ∞, it can be shown [24] that $\mu_{i\infty} = \mu_{\infty}$ in the domain D. Evaluating this equation at $x = y = 0$ and noting that

$$\mu_{i\infty}(0,0,k) = \hat{q}_1(k), \quad \mu_{\infty}(0,0,k) = \hat{q}_2(k), \quad (11.52)$$

we find the global relation (11.45).

Equation (9.28) implies the following Lax pair:

$$\mu_z - i\beta k\mu = q_z + i\beta kq, \quad \mu_{\bar{z}} - \frac{i\beta}{k}\mu = -\left(q_{\bar{z}} + \frac{i\beta}{k}q\right). \quad (11.53)$$

The first of these equations suggest that

$$\mu = -q + O\left(\frac{1}{k}\right), \quad k \to \infty, \quad (11.54)$$

which indeed can be verified for each of the functions μ_0, $\mu_{i\infty}$, μ_{∞}. Using (11.54) it is possible to formulate an RH problem for the sectionally analytic function μ, which for $z \in \Omega$ is defined by

$$\mu = \begin{cases} \mu_0, & k \in D_0, \\ \mu_{i\infty}, & k \in D_{i\infty}, \\ \mu_{\infty}, & k \in D_{\infty}. \end{cases} \quad (11.55)$$

Let S denote the following arc:

$$S = \{|k| = 1\} \cap \left\{-\frac{\pi}{2} \leq \arg k \leq 0\right\}.$$

The relevant jumps are given by

$$\mu_{i\infty} - \mu_0 = e^{i\beta\left(kz + \frac{\bar{z}}{k}\right)}\hat{q}_1(k), \quad k \in L_1 \backslash S, \quad (11.56a)$$

$$\mu_{\infty} - \mu_0 = e^{i\beta\left(kz + \frac{\bar{z}}{k}\right)}\hat{q}_2(k), \quad k \in L_2 \backslash S. \quad (11.56b)$$

$$\mu_{i\infty} - \mu_{\infty} = \hat{q}_1(k) - \hat{q}_2(k), \quad k \in S. \quad (11.56c)$$

The above expressions follow from the fact that any jump equals $\exp[i\beta(kz + \bar{z}/k]\rho(k)$, where the function $\rho(k)$ can be evaluated by letting $x = y = 0$.

The solution of the RH problem satisfying the estimate (11.54) and having a jump $J(k)$ along L is given by

$$\mu = -q + \frac{1}{2i\pi}\int_L e^{i\beta\left(lz+\frac{\bar{z}}{l}\right)} J(l)\frac{dl}{l-k}, \quad k \in \mathbb{C}\backslash L.$$

Substituting this expression in the second of equations (11.53) we find (11.42). □

11.4 From the Physical to the Spectral Plane

The novel integral representations in the spectral plane obtained earlier provide the analogue of the classical integral representations in the physical plane which can be obtained via Green's identities. Actually, as it was noted in the introduction, it is possible to derive the novel representations starting with the representations in the physical plane.

Consider as an example the modified Helmholtz equation (54) formulated in a piecewise smooth domain Ω in $\mathbb{R}^2$. Using Green's identities it follows that

$$q(x, y) = \int_{\partial\Omega} \left[(Gq_{y'} - G_{y'}q)dx' - (Gq_{x'} - G_{x'}q)dy'\right], \quad (x, y) \in \Omega, \qquad (11.57)$$

where $\partial\Omega$ denotes the boundary of Ω and $G(x, y; x', y')$ denotes the associated fundamental solution, which satisfies

$$G_{xx} + G_{yy} - 4\beta^2 G = \delta(x - x')\delta(y - y'). \qquad (11.58)$$

Using the Fourier transform representation of the δ function, i.e.,

$$\delta(x)\delta(y) = \frac{1}{4\pi^2}\int\int_{\mathbb{R}^2} e^{ik_1x + ik_2y} dk_1 dk_2, \quad (x, y) \in \mathbb{R}^2, \qquad (11.59)$$

to rewrite the RHS of (11.58) and then solving the resulting equation we find

$$G = -\frac{1}{4\pi^2}\int\int_{\mathbb{R}^2} \frac{e^{ik_1(x-x')+ik_2(y-y')}}{k_1^2 + k_2^2 + 4\beta^2} dk_1 dk_2. \qquad (11.60)$$

Substituting this formula in the RHS of (11.57) we find an expression for $q(x, y)$ which involves an integral along $\partial\Omega$ as well as integrals with respect to dk_1 and dk_2. In the case that Ω is the interior of a convex polygon, the representation in the spectral plane given by (51) involves an integral along $\partial\Omega$ and an integral with respect to dk. Thus, in order to derive this latter representation from (11.57) is it necessary to perform one integration, which turns out to be an integration with respect to dk_N where k_N is the component of the vector (k_1, k_2) which is normal to the boundary. Performing this integration for the modified Helmholtz equation is rather cumbersome; see [26] for details. However, it is straightforward to perform the analogous integration for the case of the Laplace equation provided that one uses the representation for q_z instead of the representation for q.

Example 11.7 (the Laplace equation). Let q satisfy the Laplace equation. Then q_z is an analytic function, and hence Cauchy's theorem implies

$$\frac{\partial q}{\partial z} = \frac{1}{2i\pi}\int_{\partial\Omega}\frac{\frac{\partial q}{\partial z'}dz'}{z'-z}, \quad z = x+iy \in \Omega. \tag{11.61}$$

The analogue of (11.60) is

$$\frac{1}{2i\pi}\frac{1}{z'-z} = \frac{i}{4\pi^2}\int\int_{\mathbb{R}^2}\frac{e^{ik_1(x-x')+ik_2(y-y')}}{ik_1-k_2}dk_1dk_2. \tag{11.62}$$

This equation is a direct consequence of the identity

$$\frac{1}{z} = \frac{1}{2\pi}\int\int_{\mathbb{R}^2}\frac{e^{ik_1x+ik_2y}}{ik_1-k_2}dk_1dk_2, \quad (x,y)\in\mathbb{R}^2, \tag{11.63}$$

which can be derived by integrating the identity

$$\frac{1}{\pi}\frac{\partial}{\partial\bar{z}}\left(\frac{1}{z}\right) = \delta(z), \tag{11.64}$$

with $\delta(z)$ given by the RHS of (11.59) written in terms of $(z,\bar{z})$, i.e.,

$$\delta(z) = \frac{1}{4\pi^2}\int\int_{\mathbb{R}^2}e^{ik_1\left(\frac{z+\bar{z}}{2}\right)+ik_2\left(\frac{z-\bar{z}}{2i}\right)}dk_1dk_2, \quad \bar{z} = x-iy.$$

Using (11.62) in (11.61) we find

$$q_z = \frac{i}{4\pi^2}\int\int_{\mathbb{R}^2}dk_1dk_2\int_{\partial\Omega}\frac{e^{ik_1(x-x')+ik_2(y-y')}}{ik_1-k_2}q_{z'}dz'. \tag{11.65}$$

If Ω is the interior of a convex polygon, then

$$\int_{\partial\Omega} = \sum_{j=1}^{n}\int_{z_j}^{z_{j+1}}. \tag{11.66}$$

We will now show that if $z' = x'+iy'$ is on the side (z_j, z_{j+1}), then

$$\frac{i}{4\pi^2}\int\int_{\mathbb{R}^2}dk_1dk_2\frac{e^{ik_1(x-x')+ik_2(y-y')}}{ik_1-k_2} = \frac{1}{2\pi}\int_{l_j}e^{ik(z-z')}dk, \tag{11.67}$$

and hence (11.65) and (11.66) yield (11.3).

Let e_T be the unit vector along the side (z_j, z_{j+1}) and e_N the unit vector perpendicular to this side and pointing in the outward direction. Let k_T and k_N denote the components of the vector (k_1, k_2) along e_T and e_N. Then

$$k_1 + ik_2 = (k_T - ik_N)e^{i\Theta_j}, \tag{11.68}$$

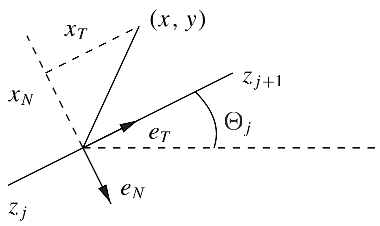


Figure 11.13. *The orthonormal vectors e_T and e_N on the side (z_j, z_{j+1}).*

where Θ_j is the angle that the side makes with the positive x-axis, see Figure 11.13. Let (X_T, X_N) denote the components of the vector $(x - x', y - y')$ along k_T and k_N. Then

$$(x - x') + i(y - y') = (X_T - iX_N)e^{i\Theta_j}. \tag{11.69}$$

Since the inner product is invariant under rotation, it follows that

$$k_1(x - x') + k_2(y - y') = k_T X_T + k_N X_N.$$

Hence the LHS of (11.67), denoted by I, becomes

$$I = \frac{i}{4\pi^2} \int \int_{\mathbb{R}^2} dk_N dk_T \frac{e^{i(k_T X_T + k_N X_N)}}{ik_T + k_N} e^{-i\Theta_j}. \tag{11.70}$$

The point (x, y) is inside the polygon and e_N is pointing in the outward direction, thus $X_N < 0$, and hence we can compute the integral with respect to dk_N in (11.70) by using Cauchy's theorem in the lower half complex k_N-plane. Thus

$$I = \frac{1}{2\pi} \int_0^\infty dk_T e^{i(k_T X_T - ik_T X_N)} e^{-i\Theta_j}.$$

Equation (11.69) implies

$$X_T - iX_N = (z - z')e^{-i\Theta_j},$$

and hence

$$I = \frac{1}{2\pi} \int_0^\infty dk_T e^{ik_T(z-z')\exp[-i\Theta_j]} e^{-i\Theta_j}.$$

Then (11.67) follows using the change of variables $k = k_T \exp[-i\Theta_j]$.

Part IV

Novel Analytical and Numerical Methods for Elliptic PDEs in a Convex Polygon

The implementation of the new transform method to linear evolution PDEs presented in Part I is based on the analysis of two fundamental equations formulated in the spectral plane: the integral representation and the global relation. The corresponding fundamental equations for the Laplace, modified Helmholtz, and Helmholtz equations in a convex polygon were derived in Part III. In Chapter 12, by utilizing these fundamental equations, we will employ the new transform method to solve several simple boundary value problems in certain polygonal domains.

A problem can be solved explicitly by the new transform method if and only if the transforms of the unknown boundary values can be eliminated from the integral representation of the solution. For complicated boundary conditions or for complicated domains this elimination is *not* possible. However, in these cases the transforms of the unknown boundary values can be characterized through the solution of certain Riemann–Hilbert (RH) problems. Several such problems will be formulated in Chapter 13.

In Chapter 14 we will introduce a novel method for computing numerically the unknown boundary values. This method can be considered as an analogue in the Fourier plane of the boundary element method (which is formulated in the physical plane).

Notations

- The given Dirichlet or Neumann boundary data will be denoted by d and n, respectively, and their transforms by D and N. Other types of given boundary data will be denoted by g (for given) and their transforms by G. The transforms of the unknown boundary values will be denoted by U (for unknown).

- If q is real, then the second global relation for the modified Helmholtz equation (i.e., the second of equations (52) of the introduction) can be obtained from the first global relation (i.e., the first of equations (52) of the introduction) by complex conjugation and then by replacing $\bar{k}$ with k. If q is complex, then the second global relation can be obtained from the first by complex conjugation of every term, *except* of those terms involving q, and then by replacing $\bar{k}$ with k. We will refer to this latter procedure as *Schwarz conjugation*. Similar considerations apply to the Laplace and Helmholtz equations.

Chapter 12

The New Transform Method for Elliptic PDEs in Simple Polygonal Domains

The implementation of the new transform method to linear evolution PDEs makes crucial use of the following facts: (a) There exist transformations in the complex k-plane which leave invariant the transforms of the boundary values. (b) The contribution of the *unknown* function $\hat{q}(k, T)$ to the integral representation either vanishes or yields a contribution which can be computed explicitly through a residue calculation. It turns out that the implementation of the new transform method to elliptic PDEs uses similar facts. Consider for example the Laplace equation in the quarter plane (see Example 11.1); for the Dirichlet problem, the unknown boundary values are given by $-iU_1/2$ and $-iU_2/2$, where

$$U_1(k) = \int_0^\infty e^{ky} q_x(0, y)dy, \quad \operatorname{Re} k \leq 0,$$

and

$$U_2(-ik) = \int_0^\infty e^{-ikx} q_y(x, 0)dx, \quad \operatorname{Im} k \leq 0.$$

It is important to recall that for second order elliptic PDEs there exist *two* global relations. The second global relation involves the Schwarz conjugate of the integrals of the boundary values, which in this example are $U_1(k)$ and $U_2(ik)$. Hence, the two global relations involve the following two *vectors* $(U_1(k), U_1(k))$, $(U_2(ik), U_2(-ik))$. Regarding fact (a) mentioned above, we note that the components of the second vector remain *invariant* under the transformation $k \to -k$. Thus, in this case we must supplement the two global relations with the two equations obtained by replacing k with $-k$. Regarding fact (b) above, we note that using the global relations and the equations obtained under the transformation $k \to -k$, it is possible to express $U_1(k)$ and $U_2(-ik)$ in terms of $U_2(ik)$. This function is analytic in the first quadrant of the complex k-plane, which is the domain involved in the integral representation (see Figure 11.4), and it turns out that its contribution vanishes.

In what follows we will illustrate this method using several examples.

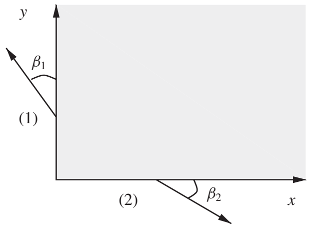


Figure 12.1. *The Laplace equation in the quarter plane.*

12.1 The Laplace Equation in the Quarter Plane

For concreteness we solve a problem with oblique Neumann boundary conditions. Several other types of boundary conditions can be treated similarly (see Remark 12.1).

Proposition 12.1. Let the complex-valued function $q(x, y)$ satisfy the Laplace equation in the quarter plane, $0 < \arg z < \frac{\pi}{2}$. Suppose that the derivative of the function q is prescribed along the direction making an angle β_1 with the y-axis and along the direction making an angle β_2 with the x-axis (see Figure 12.1),

$$-q_x(0, y)\sin\beta_1 + q_y(0, y)\cos\beta_1 = g_1(y), \quad 0 < y < \infty, \tag{12.1a}$$

$$-q_y(x, 0)\sin\beta_2 + q_x(x, 0)\cos\beta_2 = g_2(x), \quad 0 < x < \infty, \tag{12.1b}$$

where the complex-valued functions $g_1(y)$ and $g_2(x)$ have appropriate smoothness and decay and the real constants β_1, β_2 satisfy

$$0 \le \beta_1 < \pi, \quad 0 \le \beta_2 < \pi. \tag{12.2a}$$

Assume that β_1 and β_2 satisfy the condition

$$\beta_1 + \beta_2 = \frac{n\pi}{2}, \quad n = 0, 1, 2, 3. \tag{12.2b}$$

Define the following transforms of the given data:

$$G_1(k) = -\frac{1}{2}\int_0^\infty e^{ky} g_1(y)dy, \quad G_2(k) = \frac{1}{2}\int_0^\infty e^{kx} g_2(x)dx, \quad \mathrm{Re}\, k \le 0. \tag{12.3}$$

The solution $q(x, y)$ satisfies

$$\begin{aligned}\frac{\partial q}{\partial z} &= \frac{1}{\pi}\int_0^{i\infty} e^{ikz}\left[e^{-i\beta_1}G_1(k) + e^{-i(2\beta_1+\beta_2)}G_2(ik)\right]dk \\ &+ \frac{1}{\pi}\int_0^\infty e^{ikz}\left[e^{i(\beta_1+2\beta_2)}G_1(-k) + e^{i\beta_2}G_2(-ik)\right]dk, \quad 0 < \arg z < \frac{\pi}{2}.\end{aligned} \tag{12.4}$$

Proof. Let $u_1(y)$ and $u_2(x)$ denote the unknown derivatives in the directions normal to the directions of the given derivatives, i.e.,

$$-q_y(0,y)\sin\beta_1 - q_x(0,y)\cos\beta_1 = u_1(y), \quad 0<y<\infty, \tag{12.5a}$$

$$q_x(x,0)\sin\beta_2 + q_y(x,0)\cos\beta_2 = u_2(x), \quad 0<x<\infty. \tag{12.5b}$$

Solving (12.1a) and (12.5a) for $(q_x(0,y), q_y(0,y))$, as well as (12.1b) and (12.5b) for $(q_y(x,0), q_x(x,0))$ we find

$$\begin{aligned} q_y(0,y) &= g_1(y)\cos\beta_1 - u_1(y)\sin\beta_1,\\ q_x(0,y) &= -g_1(y)\sin\beta_1 - u_1(y)\cos\beta_1,\\ q_y(x,0) &= -g_2(x)\sin\beta_2 + u_2(x)\cos\beta_2,\\ q_x(x,0) &= g_2(x)\cos\beta_2 + u_2(x)\sin\beta_2. \end{aligned}$$

Substituting these expressions in the definitions of $\hat{q}_1(k)$ and $\hat{q}_2(k)$ (see (11.17)), we find

$$\hat{q}_1(k) = e^{-i\beta_1}\left[G_1(k) + iU_1(k)\right], \quad \hat{q}_2(k) = e^{i\beta_2}\left[G_2(-ik) + iU_2(-ik)\right], \tag{12.6}$$

where G_1 and G_2 are the known functions defined in (12.3), whereas U_1 and U_2 denote the transforms of the unknown functions u_1 and u_2,

$$U_1(k) = \frac{1}{2}\int_0^\infty e^{ky}u_1(y)dy, \quad U_2(k) = -\frac{1}{2}\int_0^\infty e^{kx}u_2(x)dx, \quad \operatorname{Re} k \le 0. \tag{12.7}$$

Substituting the expressions for $\hat{q}_1$ and $\hat{q}_2$ in the global relation (11.18) and also taking the Schwarz conjugate of the resulting equation, we find

$$e^{-i\beta_1}G_1(k) + ie^{-i\beta_1}U_1(k) + e^{i\beta_2}G_2(-ik) + ie^{i\beta_2}U_2(-ik) = 0, \quad \pi \le \arg k \le \frac{3\pi}{2}, \tag{12.8a}$$

$$e^{i\beta_1}G_1(k) - ie^{i\beta_1}U_1(k) + e^{-i\beta_2}G_2(ik) - ie^{-i\beta_2}U_2(ik) = 0, \quad \frac{\pi}{2} \le \arg k \le \pi. \tag{12.8b}$$

Equations (12.8) are two equations for the three unknown functions $U_1(k)$, $U_2(-ik)$, $U_2(ik)$. The pair involving U_2 remains invariant under the transformation $k \to -k$, and thus we can supplement (12.8) with the equations obtained under this transformation.

The representation for q_z (see (11.19)) involves integrals along the boundary of the first quadrant of the complex k-plane. Since in this domain the function $U_2(ik)$ is analytic, we will express $\hat{q}_1$ and $\hat{q}_2$ in terms of this function (we can also choose $U_1(-k)$ instead of $U_2(ik)$ since $U_1(-k)$ is also analytic in the first quadrant): Equation (12.8b) yields

$$iU_1(k) = -ie^{-i(\beta_1+\beta_2)}U_2(ik) + G_1(k) + e^{-i(\beta_1+\beta_2)}G_2(ik), \quad \frac{\pi}{2} \le \arg k \le \pi. \tag{12.9a}$$

In order to determine $U_2(-ik)$ we eliminate the function $U_1(k)$ from (12.8) and then we replace k by $-k$ in the resulting equation (this is of course equivalent to eliminating the function $U_1(-k)$ from the equations obtained from (12.8) under the transformation $k \to -k$)

$$iU_2(-ik) = ie^{2i(\beta_1+\beta_2)}U_2(ik) + 2e^{i(\beta_1+\beta_2)}G_1(-k) + e^{2i(\beta_1+\beta_2)}G_2(ik) + G_2(-ik),$$

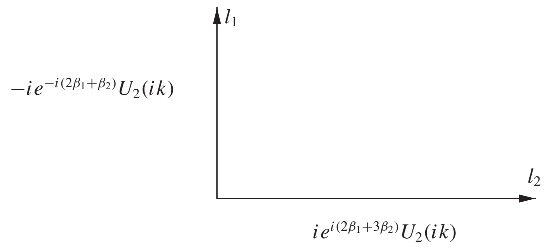


Figure 12.2. *The functions involving $U_2(ik)$.*

$$k = \mathbb{R}^+. \tag{12.9b}$$

Substituting the expressions for $iU_1(k)$ and $iU_2(-ik)$ in the formulae for $\hat{q}_1$ and $\hat{q}_2$, i.e., in (12.6), we find

$$\begin{aligned}\hat{q}_1(k) &= 2e^{-i\beta_1}G_1(k) + e^{-i(2\beta_1+\beta_2)}G_2(ik) - ie^{-i(2\beta_1+\beta_2)}U_2(ik), \quad k \in l_1,\\ \hat{q}_2(k) &= 2e^{i\beta_2}G_2(-ik) + e^{i(2\beta_1+3\beta_2)}G_2(ik) + 2e^{i(\beta_1+2\beta_2)}G_1(-k) + ie^{i(2\beta_1+3\beta_2)}U_2(ik),\\ &\qquad k \in l_2,\end{aligned} \tag{12.10}$$

where the rays l_1 and l_2 are as shown in Figure 11.4.

The terms involving $U_2(ik)$, which are shown in Figure 12.2, yield a zero contribution to q_z. Indeed, the real part of ikz equals $-k_R x - k_I y$, and thus the exponential $\exp[ikz]$ is bounded in the first quadrant of the complex k-plane. Furthermore, the function $U_2(ik)$ is analytic and of order $O(\frac{1}{k})$ as $k \to \infty$. Thus, Jordan's lemma applied to the first quadrant of the complex k-plane implies that the contribution of this unknown function vanishes, provided that

$$e^{-i(2\beta_1+\beta_2)} = e^{i(2\beta_1+3\beta_2)}, \quad \text{i.e.,} \quad e^{4i(\beta_1+\beta_2)} = 1.$$

Using Jordan's lemma in the first quadrant of the complex k-plane we can transform the contribution of the term involving $G_2(ik)$ from the integral along l_2 into the integral along l_1 and hence equations (12.10) become the expressions appearing in the integrals on the RHS of (12.4). □

12.1.1. Green's Function Representations

Replacing in (12.4) G_1 and G_2 by the expressions in (12.3) and computing explicitly the relevant k-integrals, we find

$$\begin{aligned}q_z &= \frac{1}{2\pi}\int_0^\infty \left(\frac{e^{i(\beta_1+2\beta_2)}}{iz-\xi} + \frac{e^{-i\beta_1}}{iz+\xi}\right) g_1(\xi)d\xi\\ &+ \frac{i}{2\pi}\int_0^\infty \left(\frac{e^{i\beta_2}}{z-\xi} + \frac{e^{-i(2\beta_1+\beta_2)}}{z+\xi}\right) g_2(\xi)d\xi, \quad 0 < \arg z < \frac{\pi}{2}.\end{aligned} \tag{12.11}$$

The fact that the relevant k-integrals can be computed explicitly suggests that it is possible to solve this problem by using the method of images. Indeed, using an approach similar to the one used in section I.4.2 of the introduction it is possible to derive (12.11) by utilizing the invariant properties of the global relation in the physical plane.

12.1.2. Contour Deformations

The first and the second integrals on the RHS of (12.4) involve the following exponentials:

$$e^{-|k|x-i|k|y}, \quad e^{i|k|x-|k|y},$$

as well as the following functions:

$$G_2(-|k|), \quad G_1(i|k|), \quad G_1(-|k|), \quad G_2(-i|k|).$$

The terms $\exp[-|k|x]$, $\exp[-|k|y]$, $G_2(-|k|)$, and $G_2(-|k|)$ decay exponentially, whereas the terms $\exp[-i|k|y]$, $\exp[i|k|x]$, $G_1(i|k|)$, and $G_2(-i|k|)$ oscillate as $|k| \to \infty$. Actually, using appropriate contour deformations, instead of the oscillatory exponentials $\exp[i|k|y]$ and $\exp[i|k|x]$, it is possible to obtain decaying exponentials. This is illustrated in the following example.

Example 12.1. Let

$$g_1(y) = e^{-ay}, \ \ 0 < y < \infty; \ \ g_2(x) = e^{-bx}, \ \ 0 < x < \infty, \ \ a > 0, \ \ b > 0. \tag{12.12}$$

Then, (12.4) and Cauchy's theorem imply that q_z can be expressed in terms of an integral such that $\exp[ikz]$ decays in both terms involving x and y,

$$q_z = \frac{1}{2\pi} \int_L e^{ikz} \left[\frac{e^{-i\beta_1}}{k-a} - \frac{e^{-i(\beta_1+2\beta_2)}}{k+a} - \frac{e^{-i(2\beta_1+\beta_2)}}{ik-b} + \frac{e^{i\beta_2}}{ik+b} \right], \tag{12.13}$$

where L is any ray emanating from the origin such that $0 < \arg k < \frac{\pi}{2}$. The numerical evaluation of this integral is straightforward; see Chapter 3.

Remark 12.1. A similar boundary value problem for the Poisson instead of the Laplace equation is solved in [27].

The Laplace equation in the quarter plane with oblique Robin boundary conditions, i.e., with the boundary conditions

$$-q_x(0, y) \sin\beta_1 + q_y(0, y) \cos\beta_1 + \gamma_1 q(0, y) = g_1(y), \quad 0 < y < \infty, \tag{12.14a}$$

$$-q_y(x, 0) \sin\beta_2 + q_x(x, 0) \cos\beta_2 + \gamma_2 q(x, 0) = g_2(x), \quad 0 < x < \infty, \tag{12.14b}$$

where (γ_1, γ_2) are real constants, is investigated in [29]. It is shown in [29] that the new transform method yields an explicit representation for q_z, provided that (β_1, β_2) satisfy the condition (12.2b) and the real constants (γ_1, γ_2) satisfy the condition

$$\gamma_2^2 \sin 2\beta_1 - \gamma_1^2 \sin 2\beta_2 = 0. \tag{12.15}$$

If γ_1 and/or γ_2 are different from zero, then the relevant k-integrals *cannot* be computed explicitly, which implies that the associated problems *cannot* be solved by a finite number of images.

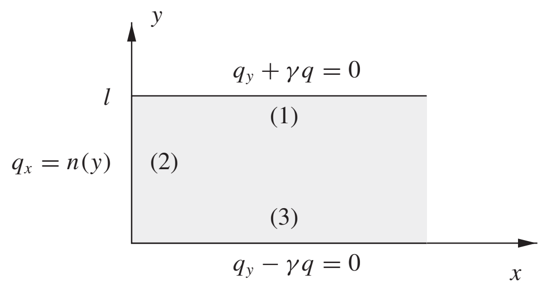


Figure 12.3. *The Laplace equation in a semi-infinite strip.*

12.2 The Laplace Equation in a Semi-Infinite Strip

Let $q(x, y)$ satisfy the Laplace equation in the semi-infinite strip Ω,

$$\Omega = \{0 < x < \infty, 0 < y < l\}, \quad l > 0. \tag{12.16}$$

For concreteness we solve a particular Robin problem. Several other types of boundary conditions can be treated similarly (see Remark 12.2).

Proposition 12.2. Let the complex-valued function $q(x, y)$ satisfy the Laplace equation in the semi-infinite strip defined in (12.16) with the boundary conditions shown in Figure 12.3, where γ is a positive constant and the complex-valued function $n(y)$ has appropriate smoothness and also satisfies the symmetry condition

$$n(y) = n(l - y), \quad 0 < y < l. \tag{12.17}$$

Define the known functions $N(k)$ and $\Gamma(k)$ by

$$N(k) = \frac{1}{2}\int_0^l e^{ky} n(y)dy, \quad \Gamma(k) = k - \gamma, \quad k \in \mathbb{C}. \tag{12.18}$$

Then the solution $q(x, y)$ satisfies

$$\begin{aligned}\frac{\partial q}{\partial z} = \frac{i}{\pi}\Bigg\{\int_0^{-\infty} e^{ikz+kl}\frac{\Gamma(-k)N(k)}{\Gamma(k)+e^{kl}\Gamma(-k)}dk - \int_0^{i\infty} e^{ikz}N(k)dk \\ + \int_0^{\infty} e^{ikz}\frac{\Gamma(k)[e^{-kl}N(k)]}{\Gamma(-k)+e^{-kl}\Gamma(k)}dk\Bigg\}, \quad z \in \Omega.\end{aligned} \tag{12.19}$$

Proof. The symmetry of the domain and of the boundary conditions implies the symmetry relation $q(x, l - y) = q(x, y)$. Hence $q(x, l) = q(x, 0)$.

Replacing $q_x(0, y)$ by $n(y)$ in the expression for $\hat{q}_2$ (see (11.20b)), we find

$$\hat{q}_2(k) = -iN(k) - U_2(k), \quad U_2(k) = \frac{1}{2}\int_0^l e^{ky} q_y(0, y)dy, \quad k \in \mathbb{C}. \tag{12.20}$$

Replacing q_y by γq in the expression for $\hat{q}_3$ (see (11.20c)), and also integrating by parts the term involving q_x, we find

$$\hat{q}_3(k) = i\Gamma(k)U_3(-ik) - q_0, \quad U_3(k) = \frac{1}{2}\int_0^\infty e^{kx}q(x,0)dx, \quad \operatorname{Re} k \le 0, \tag{12.21}$$

where $q_0 = q(0,0)/2$. Similarly, (11.20a) implies

$$\hat{q}_1(k) = e^{kl}[i\Gamma(-k)U_3(-ik) + q_0], \quad \operatorname{Im} k \le 0. \tag{12.22}$$

Using the expressions for $\{\hat{q}_j(k)\}_1^3$ in the global relation (11.21) and also taking the Schwarz conjugate of the resulting equation, we find

$$-U_2(k) + i\Delta(k)U_3(-ik) = iN(k) + \left(1 - e^{kl}\right)q_0, \quad \operatorname{Im} k \le 0, \tag{12.23a}$$

$$-U_2(k) - i\Delta(k)U_3(ik) = -iN(k) + \left(1 - e^{kl}\right)q_0, \quad \operatorname{Im} k \ge 0, \tag{12.23b}$$

where

$$\Delta(k) = \Gamma(k) + e^{kl}\Gamma(-k), \quad k \in \mathbb{C}. \tag{12.24}$$

The representation of q_z (see (11.22)) involves integrals in $\mathbb{C}^+$. Thus we will express the unknown functions $U_2(k)$ and $U_3(-ik)$ appearing in $\{\hat{q}_j\}_1^3$ in terms of the unknown function $U_3(ik)$ which is analytic in $\mathbb{C}^+$: Subtracting equations (12.23) we find

$$U_3(-ik) = -U_3(ik) + \frac{2N(k)}{\Delta(k)}, \quad k \in \mathbb{R}. \tag{12.25a}$$

Solving (12.23b) for $U_2(k)$ we find

$$-U_2(k) = i\Delta(k)U_3(ik) - iN(k) + (1 - e^{kl})q_0, \quad \operatorname{Im} k \ge 0. \tag{12.25b}$$

We note that the RHS of (12.25a) is well defined since $\Delta(k) \ne 0$ for $k \in \mathbb{R}$. Indeed, $\Delta(k) = 0$ for $k \in \mathbb{R}$ implies

$$\frac{k-\gamma}{k+\gamma} = e^{kl}, \quad k \in \mathbb{R},$$

which is a contradiction since the LHS is less than 1, whereas the RHS is greater than 1.

Substituting the expressions for $U_2(k)$ and $U_3(-ik)$ in the expressions for $\{\hat{q}_j\}_1^3$, we find

$$\hat{q}_1 = 2ie^{kl}\frac{\Gamma(-k)}{\Delta(k)}N(k) + e^{kl}[-i\Gamma(-k)U_3(ik) + q_0],$$

$$\hat{q}_2 = -2iN(k) + e^{kl}[i\Gamma(-k)U_3(ik) - q_0] + i\Gamma(k)U_3(ik) + q_0,$$

$$\hat{q}_3 = \frac{2i\Gamma(k)N(k)}{\Delta(k)} - i\Gamma(k)U_3(ik) - q_0.$$

The first terms on the RHS of these expressions yield (12.19), whereas the remaining terms (which are shown in Figure 12.4) yield a zero contribution. Indeed, the integral along $-l_3 \cup l_2$ involves the functions $\exp[ikz]$ and $i\Gamma(k)U_3(ik) + q_0$, and both these functions are bounded and analytic in the first quadrant of the complex k-plane. Furthermore,

$$i\Gamma(k)U_3(ik) + q_0 = \frac{i(k-\gamma)}{2}\int_0^\infty e^{ikx}q(x,0)dx + q_0 = O\left(\frac{1}{k}\right), \quad k \to \infty,$$

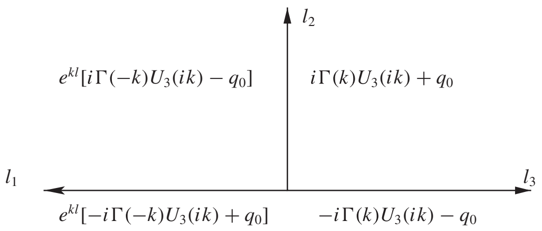


Figure 12.4. *The terms involving $U_3(ik)$.*

and thus Jordan's lemma applied in the first quadrant implies that the integral along $-l_3 \cup l_2$ vanishes. Similarly, the function $i\Gamma(-k)U_3(ik) - q_0$ is analytic in the second quadrant of the complex plane and is of order $O(1/k)$ as $k \to \infty$. Furthermore $\exp[k(iz + l)]$ is bounded in the second quadrant since

$$\left|e^{k(iz+l)}\right| = e^{-k_I x + k_R(l-y)}.$$

Hence, Jordan's lemma applied in the second quadrant implies that the integral along $-l_1 \cup l_2$ vanishes. $\square$

12.2.1. Green's Functions and Contour Deformations

Equation (12.19) can be written in the form

$$q_z = \int_0^l G(z, \xi) n(\xi) d\xi,$$

where the function $G(z, \xi)$ is defined in terms of certain k-integrals. These integrals cannot be computed in a closed form, and therefore this problem cannot be solved by a finite number of images.

The first, second, and third integrals in the RHS of (12.19) involve the following exponentials:

$$e^{-i|k|x - |k|(l-y)}, \quad e^{-|k|x - i|k|y}, \quad e^{i|k|x - |k|y},$$

as well as the functions

$$\frac{\Gamma(-|k|)N(-|k|)}{\Gamma(-|k|) + e^{-|k|l}\Gamma(|k|)}, \quad N[i|k|], \quad \frac{\Gamma(|k|)\left[e^{-|k|l}N(|k|)\right]}{\Gamma(-|k|) + e^{-|k|l}\Gamma(|k|)}.$$

The above exponentials contain terms which are either bounded or decay as $|k| \to \infty$. Actually, by using appropriate contour deformations, instead of the oscillatory exponentials $\exp[\pm i|k|x]$ and $\exp[-i|k|y]$, it is possible to obtain decaying exponentials.

Remark 12.2. The Laplace equation with the following oblique Robin boundary conditions is investigated in [29], see Figure 12.5:

$$q_x \cos\beta_1 + q_y \sin\beta_1 + \gamma_1 q = g_1(x), \quad 0 < x < \infty, \quad y = l, \tag{12.26a}$$

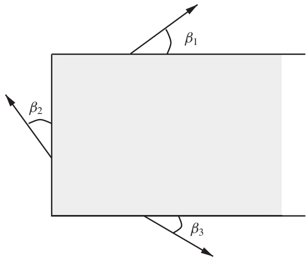


Figure 12.5. *The modified Helmholtz equation in a semi-infinite strip.*

$$q_y \cos\beta_2 - q_x \sin\beta_2 + \gamma_2 q = g_2(y), \quad x = 0, \quad 0 < y < l, \tag{12.26b}$$

$$q_x \cos\beta_3 - q_y \sin\beta_3 + \gamma_3 q = g_3(x), \quad 0 < x < \infty, \quad y = 0, \tag{12.26c}$$

where $\sin\beta_j \neq 0$, $j = 1, 2, 3$, and $(\gamma_1, \gamma_2, \gamma_3)$ are real constants. It is shown in [29] that the new transform method can be applied provided that the real constant $\{\beta_j\}_1^3$ and $\{\gamma_j\}_1^3$ satisfy the following conditions:

$$e^{4i(\beta_2-\beta_1)} = e^{4i(\beta_2+\beta_3)} = 1, \tag{12.27}$$

$$\gamma_2^2 \sin 2\beta_1 + \gamma_1^2 \sin 2\beta_2 = 0, \quad \gamma_2^2 \sin 2\beta_3 - \gamma_3^2 \sin 2\beta_2 = 0. \tag{12.28}$$

12.3 The Modified Helmholtz Equation in a Semi-Infinite Strip

For concreteness we solve the Dirichlet problem. Several other types of boundary value problems can be solved similarly (see Remark 12.3).

Proposition 12.3. Let the complex-valued function $q(x, y)$ satisfy the modified Helmholtz equation (50) in the semi-infinite strip defined in (12.16) with the Dirichlet boundary conditions

$$q(x, l) = d_1(x), \;\; q(x, 0) = d_3(x), \;\; 0 < x < \infty; \;\; q(0, y) = d_2(y), \;\; 0 < y < l. \tag{12.29}$$

Assume that the complex-valued functions $\{d_j\}_1^3$ have appropriate smoothness and are compatible at the corners $(0, 0)$, $(0, l)$ and also that the functions d_1 and d_3 have appropriate decay for large x.

Define the following transforms of the given data:

$$D_1(k) = -\int_0^\infty e^{\beta(k+\frac{1}{k})x} d_1(x)dx, \quad \mathrm{Re}\, k \leq 0,$$

$$D_2(k) = -\int_0^l e^{\beta(k+\frac{1}{k})y} d_2(y)dy, \quad k \in \mathbb{C},$$

$$D_3(k) = \int_0^\infty e^{\beta(k+\frac{1}{k})x} d_3(x)dx, \quad \mathrm{Re}\, k \le 0. \tag{12.30}$$

The solution $q(x, y)$ is given by (51a) with $n = 3$ and with the rays $\{l_j\}_1^3$ as depicted in Figure 11.6, where $\{\hat{q}_j\}_1^3$ are defined in terms of $\{D_j\}_1^3$ as follows:

$$\hat{q}_1(k) = i\beta E(k)\left[\left(k + \frac{1}{k}\right) D_1(-ik) + F_1(k)\right], \quad k \in \mathbb{R}^-, \tag{12.31a}$$

$$\hat{q}_2(k) = -i\beta\left[\left(k + \frac{1}{k}\right) E(k)D_1(ik) + 2i\left(k - \frac{1}{k}\right) D_2(k) + \left(k + \frac{1}{k}\right) D_3(ik)\right], \quad k \in i\mathbb{R}^+, \tag{12.31b}$$

$$\hat{q}_3(k) = i\beta\left[\left(k + \frac{1}{k}\right) D_3(-ik) + F_3(k)\right], \quad k \in \mathbb{R}^+, \tag{12.31c}$$

where

$$E(k) = e^{\beta(k+\frac{1}{k})l}, \quad k \in \mathbb{C}, \tag{12.31d}$$

and F_1, F_2 are defined for $k \in \mathbb{R}$ by

$$\begin{aligned} F_1(k) = -\frac{1}{E(k) - E(-k)} \Bigg\{&\left(k + \frac{1}{k}\right)\Big[E(k) + E(-k)\Big]\Big[D_1(-ik) - D_1(ik)\Big] \\ &+ 2i\left(\frac{1}{k} - k\right)\Big[D_2(-k) - D_2(k)\Big] + 2\left(k + \frac{1}{k}\right)\Big[D_3(-ik) - D_3(ik)\Big]\Bigg\}, \end{aligned} \tag{12.31e}$$

$$\begin{aligned} F_3(k) = \frac{1}{E(k) - E(-k)} \Bigg\{&2\left(k + \frac{1}{k}\right)\Big[D_1(-ik) - D_1(ik)\Big] \\ &+ 2i\left(\frac{1}{k} - k\right)\Big[E(k)D_2(-k) - E(-k)D_2(k)\Big] \\ &+ \left(k + \frac{1}{k}\right)\Big[E(k) + E(-k)\Big]\Big[D_3(-ik) - D_3(ik)\Big]\Bigg\}. \end{aligned} \tag{12.31f}$$

Proof. Replacing the Dirichlet boundary values by the given data in the expressions for $\{\hat{q}_j\}_1^3$ (see (11.40)), we find

$$\hat{q}_1(k) = E(k)\left[iU_1(-ik) + i\beta\left(k + \frac{1}{k}\right) D_1(-ik)\right], \quad \mathrm{Im}\, k \le 0,$$

$$\hat{q}_2(k) = iU_2(k) + \beta\left(\frac{1}{k} - k\right) D_2(k), \quad k \in \mathbb{C},$$

$$\hat{q}_3(k) = iU_3(-ik) + i\beta\left(k + \frac{1}{k}\right) D_3(-ik), \quad \mathrm{Im}\, k \le 0, \tag{12.32}$$

where the unknown functions $\{U_j\}_1^3$ denote the transforms of the unknown Neumann boundary values, i.e.,

$$U_1(k) = \int_0^\infty e^{\beta(k+\frac{1}{k})x} q_y(x,l)dx, \quad \operatorname{Re} k \le 0,$$

$$U_2(k) = -\int_0^l e^{\beta(k+\frac{1}{k})y} q_x(0,y)dy, \quad k \in \mathbb{C},$$

$$U_3(k) = -\int_0^\infty e^{\beta(k+\frac{1}{k})x} q_y(x,0)dx, \quad \operatorname{Re} k \le 0. \tag{12.33}$$

The global relation (11.21) yields

$$E(k)U_1(-ik) + U_2(k) + U_3(-ik) = J(k), \quad \operatorname{Im} k \le 0, \tag{12.34a}$$

where the known function $J(k)$ is defined by

$$\begin{aligned} J(k) = {} & -\beta\left(k+\frac{1}{k}\right)E(k)D_1(-ik) + i\beta\left(\frac{1}{k}-k\right)D_2(k) \\ & -\beta\left(k+\frac{1}{k}\right)D_3(-ik), \ \operatorname{Im} k \le 0. \end{aligned} \tag{12.34b}$$

The Schwarz conjugate of (12.34a) yields

$$E(k)U_1(ik) + U_2(k) + U_3(ik) = \overline{J(\bar{k})}, \quad \operatorname{Im} k \ge 0, \tag{12.34c}$$

where $\overline{J(\bar{k})}$ denotes the function obtained from $J(k)$ by taking the complex conjugate of each term of $J(k)$ *except* the terms $\{d_j\}_1^3$. The integral representation of q involves integrals in $\mathbb{C}^+$, and thus we will express the unknown functions $U_1(-ik)$, $U_2(k)$, $U_3(-ik)$ appearing in (12.32) in terms of the functions $U_1(ik)$ and $U_3(ik)$ which are analytic in $\mathbb{C}^+$. In this respect we note that (12.34c) immediately implies $U_2(k)$ in terms of $U_1(ik)$ and $U_3(ik)$. In order to express $U_1(-ik)$ and $U_3(-ik)$ in terms of $U_1(ik)$ and $U_3(ik)$ we subtract equations (12.34a) and (12.34c):

$$E(k)\left[U_1(-ik) - U_1(ik)\right] + \left[U_3(-ik) - U_3(ik)\right] = J(k) - \overline{J(\bar{k})}, \quad k \in \mathbb{R}. \tag{12.35}$$

Replacing k by $-k$ in this equation we obtain a second equation involving the same brackets that appear in (12.35). Solving this equation and (12.35) for these two unknown brackets, we find

$$U_1(-ik) = U_1(ik) + \beta F_1(k), \quad k \in \mathbb{R},$$

$$U_3(-ik) = U_3(ik) + \beta F_3(k), \quad k \in \mathbb{R}, \tag{12.36}$$

where the known functions F_1 and F_3 are defined in (12.31e) and (12.31f). Substituting in (12.32) the expressions for $U_1(-ik)$, $U_2(k)$, $U_3(-ik)$, we find

$$\hat{q}_1(k) = i\beta\left(k+\frac{1}{k}\right)E(k)D_1(-ik) + i\beta E(k)F_1(k) + iE(k)U_1(ik), \quad k \in \mathbb{R},$$

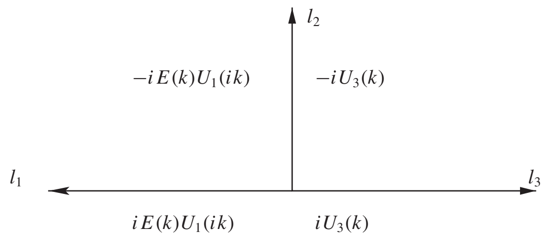


Figure 12.6. *The terms involving $U_3(k)$ and $U_1(ik)$.*

$$\hat{q}_2(k) = \beta\left(\frac{1}{k} - k\right) D_2(k) + i\overline{J(\bar{k})} - iE(k)U_1(ik) - iU_3(k), \quad \operatorname{Im} k \geq 0,$$

$$\hat{q}_3(k) = i\beta\left(k + \frac{1}{k}\right) D_3(-ik) + i\beta F_3(k) + iU_3(ik), \quad k \in \mathbb{R}. \tag{12.37}$$

The first two terms in the RHS of (12.37) yield (12.31a)–(12.31c), whereas the remaining terms, which are shown in Figure 12.6, yield a zero contribution. Indeed, the integral along $-l_3 \cup l_2$ involves the function $\exp[i\beta(kz - \bar{z}/k)]/k$ which is bounded as $k \to 0$ and as $k \to \infty$ in the first quadrant of the complex k-plane, and the function $U_3(ik)$ which is analytic and of order $O(1/k)$ as $k \to \infty$ and of order $O(k)$ as $k \to 0$; hence this integral vanishes. Similar considerations are valid for the integral along $-l_3 \cup l_1$, since the relevant exponential satisfies

$$\left| e^{i\beta\left(kz - \frac{\bar{z}}{k}\right) + \beta\left(k + \frac{1}{k}\right)l} \right| = e^{\beta\left(1 + \frac{1}{|k|^2}\right)[-k_I x + k_R(l-y)]}. \quad \square$$

12.3.1. Contour Deformations

By considering the term $E(k)$ appearing in $\hat{q}_1(k)$ together with $\exp[i\beta(kz - \bar{z}/k)]$, it follows that the representation for q contains the following functions on the contours $(0, -\infty)$, $(0, i\infty)$, and $(0, \infty)$, respectively:

$$-\frac{1}{|k|} e^{i\beta\left(\frac{1}{|k|} - |k|\right)x - \beta\left(|k| + \frac{1}{|k|}\right)(l-y)}, \quad \frac{1}{i|k|} e^{-\beta\left(\frac{1}{|k|} + |k|\right)x + i\beta\left(\frac{1}{|k|} - |k|\right)y},$$

$$\frac{1}{|k|} e^{-i\beta\left(\frac{1}{|k|} - |k|\right)x - \beta\left(|k| + \frac{1}{|k|}\right)y}.$$

Each of these terms contains a function which decays exponentially as $|k| \to \infty$ or $|k| \to 0$. In addition, the representation for q contains the functions

$$i\beta\left(k + \frac{1}{k}\right) D_1(-ik) + i\beta F_1(k), \quad \hat{q}_2(k), \quad \hat{q}_3(k),$$

which either are bounded or they decay. Actually, by using appropriate contour deformations it makes it possible to obtain decaying instead of oscillatory exponentials. This is illustrated in the following example.

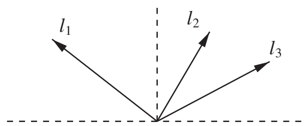


Figure 12.7. *The contours* $\{l_i\}_1^3$ *for Example* 12.2.

Example 12.2. Let

$$d_1 = d_2 = 0, \quad d_3 = xe^{-ax}, \quad 0 < x < \infty, \quad a > 0.$$

Then

$$D_1 = D_2 = 0, \quad D_3(ik) = \frac{1}{\left[i\beta\left(k - \frac{1}{k}\right) - a\right]^2}.$$

Hence

$$\hat{q}_1(k) = E(k)\tilde{q}_1(k), \quad \tilde{q}_1(k) = -\frac{2i\beta\left(k + \frac{1}{k}\right)}{E(k) - E(-k)}\Big[D_3(-ik) - D_3(ik)\Big],$$

$$\hat{q}_2(k) = i\beta\left(k + \frac{1}{k}\right) D_3(ik),$$

$$\hat{q}_3(k) = i\beta\left(k + \frac{1}{k}\right)\left\{D_3(-ik) + \frac{\Big[E(k) + E(-k)\Big]}{E(k) - E(-k)}\Big[D_3(-ik) - D_3(ik)\Big]\right\}. \quad (12.38a)$$

Thus

$$\begin{aligned} q(x,y) = \frac{1}{2\pi}\Bigg\{&\int_{L_1} e^{i\beta\left(kz - \frac{\bar{z}}{k}\right) + l\beta\left(k + \frac{1}{k}\right)}\tilde{q}_1(k)\frac{dk}{k} + \int_{L_2} e^{i\beta\left(kz - \frac{\bar{z}}{k}\right)}\hat{q}_2(k)\frac{dk}{k} \\ &+ \int_{L_3} e^{i\beta\left(kz - \frac{\bar{z}}{k}\right)}\hat{q}_3(k)\frac{dk}{k}\Bigg\}, \end{aligned} \quad (12.38b)$$

where the contours $\{L_j\}_1^3$, depicted in Figure 12.7, are defined by the requirement that $\arg k$ is in the following open intervals:

$$\left(\frac{\pi}{2}, \pi\right), \quad \left(0, \frac{\pi}{2}\right), \quad \left(0, \frac{\pi}{2}\right). \quad (12.38c)$$

The numerical evaluation of the RHS of (12.38b) is straightforward; see Chapter 3.

Remark 12.3. The modified Helmholtz equation with the oblique Robin boundary conditions defined by (12.26) is investigated in [30], where it is shown that the new transform method yields an explicit solution provided that the real constants $\{\beta_j\}_1^3$ and $\{\gamma_j\}_1^3$ satisfy conditions (12.27) as well as the following conditions:

$$(2\beta^2 - \gamma_2^2)\sin 2\beta_1 + (2\beta^2 - \gamma_1^2)\sin 2\beta_2 = 0,$$

$$(2\beta^2 - \gamma_2^2)\sin 2\beta_3 - (2\beta^2 - \gamma_3^2)\sin 2\beta_2 = 0. \quad (12.39)$$

If $\beta = 0$, these equations become (12.28). Requiring that $0 < \beta_j < \pi$ and $\gamma_j \geq 0$, $j = 1, 2, 3$, (12.27) and (12.39) yield

$$\beta_3 + \beta_2 = \frac{\pi}{2}m, \quad 2\beta^2 - \gamma_2^2 = (2\beta^2 - \gamma_3^2)(-1)^{m-1}, \quad m = 1, 2, 3,$$

$$\beta_2 - \beta_1 = \frac{\pi}{2}n, \quad 2\beta^2 - \gamma_2^2 = (2\beta^2 - \gamma_1^2)(-1)^{n-1}, \quad n = -1, 0, 1. \tag{12.40}$$

12.4 The Helmholtz Equation in the Quarter Plane

For concreteness we consider the Dirichlet problem. Other boundary conditions can be treated similarly.

Proposition 12.4. Let the complex-valued function $q(x, y)$ satisfy the Helmholtz equation (9.27) in the quarter plane, $0 < \arg z < \pi/2$, with the usual radiation condition at ∞, and with the Dirichlet boundary conditions

$$q(0, y) = d_1(y), \ \ 0 < y < \infty; \ \ q(x, 0) = d_2(x), \ \ 0 < x < \infty, \tag{12.41}$$

where the complex-valued functions d_1 and d_2 have appropriate smoothness and decay and are compatible at $x = y = 0$, i.e., $d_1(0) = d_2(0)$.

Define the following transforms of the given data:

$$D_1(k) = \int_0^\infty e^{\beta(k-\frac{1}{k})y} d_1(y)dy, \ \ D_2(k) = \int_0^\infty e^{\beta(k-\frac{1}{k})x} d_2(x)dx, \ \ \mathrm{Re}\, k \leq 0. \tag{12.42}$$

The solution $q(x, y)$ is given by (11.42), where the contours $\{L_j\}_1^2$ are as depicted in Figure 11.10 and the functions $\{\hat{q}_1\}_1^2$ are defined as follows:

$$\hat{q}_1 = -2\beta \left[i \left(k - \frac{1}{k} \right) D_2(ik) - \left(k + \frac{1}{k} \right) D_1(k) \right], \quad k \in L_1,$$

$$\hat{q}_2 = -2\beta \left[i \left(k - \frac{1}{k} \right) D_2(-ik) - \left(k + \frac{1}{k} \right) D_1(-k) \right], \quad k \in L_2. \tag{12.43}$$

Proof. According to Proposition 11.2, the solution q is given by (11.42), where $\{\hat{q}_j\}_1^2$ are defined by (11.43). Let $\{U_j\}_1^2$ denote the transforms of the unknown Neumann boundary values, i.e.,

$$U_1(k) = \int_0^\infty e^{\beta(k-\frac{1}{k})y} q_x(0, y)dy,$$

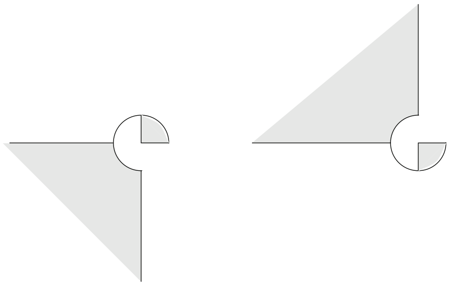
$$U_2(k) = \int_0^\infty e^{\beta(k-\frac{1}{k})x} q_y(x, 0)dx, \quad k \in D_{i\infty}. \tag{12.44}$$

Then, equations (11.43) yield

$$\hat{q}_1(k) = -iU_1(k) + \beta \left(k + \frac{1}{k} \right) D_1(k), \quad k \in D_{i\infty},$$

$$\hat{q}_2(k) = iU_2(-ik) - i\beta \left(k - \frac{1}{k} \right) D_2(-ik), \quad k \in D_\infty, \tag{12.45}$$

where the domains $D_{i\infty}$ and D_∞ are as depicted in Figures 11.9(a) and (b).



D (a) $\hat{D}$ (b)

Figure 12.8. *The domains D and $\hat{D}$.*

Ω_1 (a) Ω_2 (b)

Figure 12.9. *The domains Ω_1 and Ω_2.*

Substituting the above expressions for $\hat{q}_1$ and $\hat{q}_2$ in the global relation (11.45), we find

$$-iU_1(k) + \beta\left(k + \frac{1}{k}\right)D_1(k) = iU_2(-ik) - i\beta\left(k - \frac{1}{k}\right)D_2(-ik), \quad k \in D, \quad (12.46a)$$

where D is depicted in Figure 12.8(a). The Schwarz conjugate of this equation implies

$$iU_1(k) + \beta\left(k + \frac{1}{k}\right)D_1(k) = -iU_2(ik) + i\beta\left(k - \frac{1}{k}\right)D_2(ik), \quad k \in \hat{D}, \quad (12.46b)$$

where $\hat{D}$ is obtained from $\hat{D}$ via the transformation $k \to \bar{k}$; see Figure 12.8(b).

We will now use (12.46) to eliminate U_1 and U_2 from the integral representation of q. In this respect, we first note that in the domain where both equations (12.46) are valid, by adding these equations we find

$$2\beta\left(k + \frac{1}{k}\right)D_1(k) = iU_2(-ik) - iU_2(ik) + i\beta\left(k - \frac{1}{k}\right)\Big[D_2(ik) - D_2(-ik)\Big],$$
$$k \in \Omega_1, \quad (12.47a)$$

where the domain Ω_1 is depicted in Figure 12.9(a). Replacing k by $-k$ in this equation, we find

$$-2\beta\left(k + \frac{1}{k}\right)D_1(-k) = iU_2(ik) - iU_2(-ik) + i\beta\left(k - \frac{1}{k}\right)\Big[D_2(ik) - D_2(-ik)\Big],$$
$$k \in \Omega_2, \quad (12.47b)$$

where the domain Ω_2 is depicted in Figure 12.9(b). The unknown function $-iU_1(k)$ occurs on L_1, and since this curve belongs to the domain $\hat{D}$, we can use the global relation (12.46b) to express $-iU_1(k)$ in terms of $iU_2(ik)$; thus

$$\hat{q}_1(k) = 2\beta\left(k+\frac{1}{k}\right)D_1(k) - i\beta\left(k-\frac{1}{k}\right)D_2(ik) + iU_2(ik), \quad k \in L_1. \tag{12.48a}$$

The unknown function $iU_2(-ik)$ occurs on L_2, and since this curve belongs to the domain Ω_2, we can use (12.47b) to express $iU_2(-ik)$ in terms of $iU_2(ik)$; thus

$$\begin{aligned}\hat{q}_2(k) = &-2i\beta\left(k-\frac{1}{k}\right)D_2(-ik) + i\beta\left(k-\frac{1}{k}\right)D_2(ik)\\ &+2\beta\left(k+\frac{1}{k}\right)D_1(-k) + iU_2(ik), \quad k \in L_2.\end{aligned} \tag{12.48b}$$

By using Cauchy's theorem in the domains bounded by the parts of $L_1 \cup L_2$ in $\mathbb{C}^+$ and $\mathbb{C}^-$ we find that the contribution due to $U_2(ik)$ vanishes. Furthermore, we can transform the term involving $D_2(ik)$ from L_2 to L_1, and hence (12.48) yield (12.43). □

12.4.1. Contour Deformations

The representation (11.42) contains the following exponentials on the unbounded parts of the contours L_1 and L_2:

$$e^{-\beta\left(|k|-\frac{1}{|k|}\right)x - i\beta\left(|k|+\frac{1}{|k|}\right)y}, \quad e^{i\beta\left(|k|+\frac{1}{|k|}\right)x - \beta\left(|k|-\frac{1}{|k|}\right)y}. \tag{12.49}$$

Also, it contains the following functions on the bounded parts of the contours L_1 and L_2:

$$\frac{1}{k}e^{-\beta\left(\frac{1}{|k|}-|k|\right)x + i\beta\left(|k|+\frac{1}{|k|}\right)y}, \quad \frac{1}{k}e^{-i\beta\left(|k|+\frac{1}{|k|}\right)x - \beta\left(\frac{1}{|k|}-|k|\right)y}. \tag{12.50}$$

The exponentials in (12.49) and the functions in (12.50) contain terms which decay as $|k| \to \infty$ and $|k| \to 0$, respectively. Furthermore, the functions $\big(D_2(ik), D_1(k)\big)$ and $\big(D_2(-ik), D_1(-k)\big)$ decay on L_1 and L_2, respectively. Actually, by using appropriate contour deformations it is possible to obtain decaying instead of oscillatory exponentials.

12.5 The Modified Helmholtz Equation in an Equilateral Triangle

Proposition 12.5. Let the complex-valued function $q(x, y)$ satisfy the modified Helmholtz equation (50) in the interior of the equilateral triangle described in Example 11.6; see Figure 11.7. Assume that the same smooth complex-valued function $d(s)$ is prescribed on each side as the Dirichlet boundary condition, i.e.,

$$q^{(j)}(s) = d(s), \quad s \in \left(-\frac{l}{2}, \frac{l}{2}\right), \quad j = 1, 2, 3, \tag{12.51}$$

where each side of the triangle is parametrized in terms of s by (11.23). The solution $q(x, y)$ is given by (71).

Proof. Using the parametrizations defined by (11.23), it was shown in Example 11.6 that $\{\hat{q}_j\}_1^3$ are given by (11.41). Then, the boundary conditions (12.51) immediately imply (67), i.e.,

$$\hat{q}_1(k) = \hat{q}(k), \quad \hat{q}_2(k) = \hat{q}(\alpha k), \quad \hat{q}_3(k) = \hat{q}(\bar{\alpha} k), \tag{12.52a}$$

with

$$\hat{q}(k) = E(-ik)\left[iU(k) + \beta\left(\frac{1}{k} - k\right)D(k)\right]. \tag{12.52b}$$

Substituting these expressions in the first of the global relations, i.e., the first of equations (69), multiplying the resulting equation by $E(i\bar{\alpha}k)$, and using the identities

$$i(\bar{\alpha} - \alpha) = \sqrt{3}, \quad i(\alpha - 1) = \sqrt{3}\bar{\alpha}, \tag{12.53}$$

the first of the global relations becomes

$$e(-\alpha k)U(k) + e(k)U(\alpha k) + U(\bar{\alpha}k) = i\beta J(k), \quad k \in \mathbb{C}, \tag{12.54}$$

where $e(k) = e^{\frac{\beta l}{2}(k + \frac{1}{k})}$ and

$$\begin{aligned} J(k) = &\left(\frac{1}{k} - k\right)e(-\alpha k)D(k) + e(k)\left(\frac{1}{\alpha k} - \alpha k\right)D(\alpha k) \\ &+ \left(\frac{1}{\bar{\alpha}k} - \bar{\alpha}k\right)D(\bar{\alpha}k). \end{aligned} \tag{12.55}$$

Taking the Schwarz conjugate of the global relation (12.54) and multiplying the resulting equation by $e(-k)$, we find

$$e(\alpha k)U(k) + e(-k)U(\alpha k) + U(\bar{\alpha}k) = -i\beta e(-k)\overline{J(\bar{k})}, \quad k \in \mathbb{C}, \tag{12.56}$$

where we have used the identity

$$1 + \alpha + \bar{\alpha} = 0, \tag{12.57}$$

and $\overline{J(\bar{k})}$ denote the function obtained from $J(k)$ by taking the complex conjugate of each term of $J(k)$ except $d(s)$. Subtracting equations (12.54) and (12.56) we find the following equation which is valid for all $k \in \mathbb{C}$:

$$[e(\alpha k) - e(-\alpha k)]\,U(k) = [e(k) - e(-k)]\,U(\alpha k) - i\beta\left[J(k) + e(-k)\overline{J(\bar{k})}\right]. \tag{12.58}$$

Replacing in the equation for $\hat{q}$ (see (12.52b)), the expression for $U(k)$ obtained from (12.58) and using the identity

$$E(-ik)[e(k) - e(-k)] = E^2(i\bar{\alpha}k) - E^2(i\alpha k),$$

we find

$$\hat{q}(k) = \beta\left(\frac{1}{k} - k\right)E(-ik)D(k) + \frac{\beta G(k)E(-ik)}{\Delta(\alpha k)} + i\left[E^2(i\bar{\alpha}k) - E^2(i\alpha k)\right]\frac{U(\alpha k)}{\Delta(\alpha k)}, \tag{12.59}$$

where $\Delta(k)$ and $G(k)$ are defined, for all $k \in \mathbb{C}$, by

$$\Delta(k) = e(k) - e(-k), \quad G(k) = J(k) + e(-k)\overline{J(\bar{k})}. \tag{12.60}$$

The functions $\hat{q}_2(k)$ and $\hat{q}_3(k)$ can be obtained from the RHS of (12.59) by replacing k with αk and $\bar{\alpha} k$.

In what follows we will show that the contribution to the solution q of the unknown functions $U(\alpha k)$, $U(\bar{\alpha} k)$, $U(k)$ can be computed in terms of the given boundary conditions. In this respect we will use the following facts.

(a) The zeros of the functions $\Delta(k)$, $\Delta(\alpha k)$, $\Delta(\bar{\alpha} k)$ occur on the following lines, respectively, in the complex k-plane:

$$i\mathbb{R}, \quad e^{\frac{5i\pi}{6}}\mathbb{R}, \quad e^{\frac{i\pi}{6}}\mathbb{R}. \tag{12.61}$$

Indeed, the zeros of $\Delta(k)$ occur on the imaginary axis, and then the zeros of $\Delta(\alpha k)$ and $\Delta(\bar{\alpha} k)$ can be obtained by appropriate rotations.

(b) The functions

$$e^{i\beta\left(kz - \frac{\bar{z}}{k}\right)} E^2(i\alpha k), \quad e^{i\beta\left(kz - \frac{\bar{z}}{k}\right)} E^2(ik), \quad e^{i\beta\left(kz - \frac{\bar{z}}{k}\right)} E^2(i\bar{\alpha} k), \tag{12.62a}$$

with z in the interior of the triangle, are bounded as $k \to 0$ and $k \to \infty$, for $\arg k$ in

$$\left[-\frac{\pi}{2}, \frac{\pi}{6}\right], \quad \left[\frac{\pi}{6}, \frac{5\pi}{6}\right], \quad \left[\frac{5\pi}{6}, \frac{3\pi}{2}\right], \tag{12.62b}$$

respectively; see Figure 12.10. Indeed, let us consider the first exponential in (12.62a). Using $z_1 = -l\alpha/\sqrt{3}$, this exponential can be written as

$$e^{i\beta k(z - z_1) + \frac{\beta(\bar{z} - \bar{z}_1)}{ik}}.$$

If z is in the interior of the triangle, then

$$\frac{\pi}{2} \leq \arg(z - z_1) \leq \frac{5\pi}{6},$$

and thus, if

$$-\frac{\pi}{2} \leq \arg k \leq \frac{\pi}{6},$$

it follows that

$$0 \leq \arg\left[k(z - z_1)\right] \leq \pi.$$

Hence, the exponentials

$$e^{i\beta k(z - z_1)} \text{ and } e^{\frac{\beta(\bar{z} - \bar{z}_1)}{ik}}$$

are bounded as $|k| \to \infty$ and $|k| \to 0$, respectively. The analogous results for the second and third exponentials in (12.62a) can be obtained in a similar way.

(c) The functions

$$\frac{U(k)}{\Delta(k)}, \quad \frac{U(\alpha k)}{\Delta(\alpha k)}, \quad \frac{U(\bar{\alpha} k)}{\Delta(\bar{\alpha} k)}$$

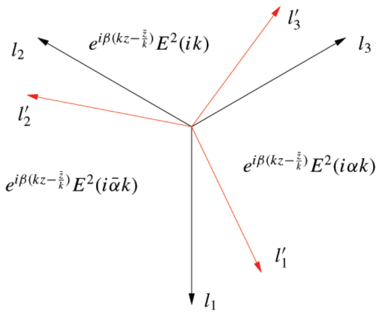


Figure 12.10. *The domains of boundedness of the function defined in* (12.62a).

are bounded in the entire complex k-plane except on the lines defined in (12.61), where the functions $\Delta(k)$, $\Delta(\alpha k)$, $\Delta(\bar{\alpha}k)$ have simple zeros.

Indeed, regarding $U(k)/\Delta(k)$ we note that $\Delta(k)$ is dominated by $e(k)$ for $\operatorname{Re} k > 0$ and by $e(-k)$ for $\operatorname{Re} k < 0$, and hence

$$\frac{U(k)}{\Delta(k)} \sim \begin{cases} e(-k)U(k), & \operatorname{Re} k > 0, \\ -e(k)U(k), & \operatorname{Re} k < 0. \end{cases}$$

Furthermore, $e(-k)U(k)$ involves $k(s - l/2)$, which is bounded for $\operatorname{Re} k \geq 0$, whereas $e(k)U(k)$ involves $k(s + l/2)$, which is bounded for $\operatorname{Re} k \leq 0$. Similar considerations are valid for the terms involving $1/k$.

The unknown function $U(\alpha k)$ in the expression for $\hat{q}(k)$ (see (12.59)) yields the following contribution $C_1(x, y)$ to the solution q:

$$C_1 = \frac{1}{4\pi}\int_{l_1} P[E^2(i\bar{\alpha}k) - E^2(i\alpha k)]\frac{U(\alpha k)}{\Delta(\alpha k)}\frac{dk}{k},$$

where P denotes the exponential

$$P = e^{i\beta\left(kz-\frac{\bar{z}}{k}\right)}.$$

The integral of the second term in the RHS of C_1 can be deformed from l_1 to l_1', where l_1' is a ray with $-\pi/2 < \arg k < -\pi/6$. Hence,

$$C_1 = \frac{1}{4\pi}\int_{l_1} PE^2(i\bar{\alpha}k)\frac{U(\alpha k)}{\Delta(\alpha k)}\frac{dk}{k} - \frac{1}{4\pi}\int_{l_1'} PE^2(i\alpha k)\frac{U(\alpha k)}{\Delta(\alpha k)}\frac{dk}{k}.$$

In the second integral on the RHS of this equation we use (12.58), i.e., the equation

$$\Delta(\alpha k)U(k) = \Delta(k)U(\alpha k) - i\beta G(k),$$

to replace $U(\alpha k)$. Hence, $C_1 = \tilde{C}_1 + \tilde{U}_1$, where

$$\tilde{U}_1 = \frac{1}{4\pi}\int_{l_1} PE^2(i\bar{\alpha}k)\frac{U(\alpha k)}{\Delta(\alpha k)}\frac{dk}{k} - \frac{1}{4\pi}\int_{l_1'} PE^2(i\alpha k)\frac{U(k)}{\Delta(k)}\frac{dk}{k} \tag{12.63a}$$

and

$$\tilde{C}_1 = \frac{\beta}{4\pi i} \int_{l_1'} P E^2(i\alpha k) \frac{G(k)}{\Delta(k)\Delta(\alpha k)} \frac{dk}{k}. \tag{12.63b}$$

In summary, the term $\hat{q}(k)$ gives rise to the contribution $F_1 + \tilde{U}_1$, where $\tilde{U}_1$ is defined in (12.63a) and F_1 is defined by

$$\begin{aligned} F_1 = &\frac{1}{4\pi i} \int_{l_1} P \left[\beta \left(\frac{1}{k} - k \right) E(-ik) D(k) + \frac{\beta E(-ik) G(k)}{\Delta(\alpha k)} \right] \frac{dk}{k} \\ &+ \frac{\beta}{4\pi i} \int_{l_1'} P E^2(i\alpha k) \frac{G(k)}{\Delta(k)\Delta(\alpha k)} \frac{dk}{k}. \end{aligned} \tag{12.64}$$

The contributions of $\hat{q}_2$ and $\hat{q}_3$ can be obtained from F_1 and $\tilde{U}_1$ using the substitutions

$$l_1 \to l_2 \to l_3, \quad l_1' \to l_2' \to l_3', \quad k \to \alpha k \to \bar{\alpha} k. \tag{12.65}$$

We will now show that the contributions of $\tilde{U}_j$, $j = 1, 2, 3$, vanish. Indeed, the integrands

$$P E^2(i\bar{\alpha} k) \frac{U(\alpha k)}{k \Delta(\alpha k)}, \quad P E^2(ik) \frac{U(\bar{\alpha} k)}{k \Delta(\bar{\alpha} k)}, \quad P E^2(i\alpha k) \frac{U(k)}{k \Delta(k)}$$

occur on $l_1 \cup l_2'$, $l_2 \cup l_3'$, $l_3 \cup l_1'$ and in the domains bounded by these contours the above functions are bounded and analytic; see Figure 12.10.

Hence,

$$q = F_1 + F_2 + F_3, \tag{12.66}$$

where F_2 and F_3 are obtained from F_1 using the substitutions (12.65).

We will now show that (12.66) is equivalent to (71a). We make the change of variables $k \to \bar{\alpha} k$ and $k \to \alpha k$ in the integrals defining F_2 and F_3, respectively. Regarding F_2 we note that under this transformation (a) the fraction dk/k remains invariant; (b) the rays l_2 and l_2' are mapped to the rays l_1 and l_1', respectively; (c) the exponential $\exp[i\beta(kz - \bar{z}/k)]$ is mapped to $\exp[i\beta(\bar{\alpha} kz - \bar{z}/\bar{\alpha} k]$; and (d) the remaining terms in the integrand of F_2 are identical to the corresponding terms of the integrand of F_2. Similar considerations are valid for F_3. □

Remark 12.4. The integrands appearing in the integrals along l_1 and l_1' defined in (71) contain terms which decay exponentially. Indeed, regarding the integral along l_1' we note that $G(k)/\Delta(k)\Delta(\alpha k)$ is bounded for k on l_1', and the function $A(k, z, \bar{z}) E^2(i\alpha k)$ contains terms which decay exponentially since each of the three terms of this function contains exponentials with negative real parts. Regarding the integral along l_1 we note that the function $D(k)$ is bounded for k on l_1, $G(k)/\Delta(k)$ decays exponentially since $s \in (-l/2, l/2)$, and each of the real terms of the function $A(k, z, \bar{z}) E(-ik)$ has an exponential with negative real part.

Example 12.3. Suppose that l and $d(s)$ are given by

$$l = \pi, \quad d(s) = \cos s. \tag{12.67}$$

Then the definitions of $D(k)$ and $G(k)$ (see (68) and (71c)) imply

$$D(s) = \frac{2}{1+\beta^2\left(k+\frac{1}{k}\right)^2}\cosh\left[\beta\left(k+\frac{1}{k}\right)\frac{\pi}{2}\right] \tag{12.68a}$$

and

$$G(k) = 4\left[\frac{\frac{1}{k}-k}{1+\beta^2\left(k+\frac{1}{k}\right)^2} + \frac{\frac{1}{\alpha k}-\alpha k}{1+\beta^2\left(\alpha k+\frac{1}{\alpha k}\right)^2}\right]\cosh\left[\beta\left(k+\frac{1}{k}\right)\frac{\pi}{2}\right]$$

$$\times\cosh\left[\beta\left(\alpha k+\frac{1}{\alpha k}\right)\frac{\pi}{2}\right] + 4\frac{\frac{1}{\bar{\alpha}k}-\bar{\alpha}k}{1+\beta^2\left(\bar{\alpha}k+\frac{1}{\bar{\alpha}k}\right)^2}\cosh\left[\beta\left(\bar{\alpha}k+\frac{1}{\bar{\alpha}k}\right)\frac{\pi}{2}\right]. \tag{12.68b}$$

Furthermore,

$$\Delta^+(k) = 2\cosh\left[\beta\left(k+\frac{1}{k}\right)\frac{\pi}{2}\right], \quad \Delta(k) = 2\sinh\left[\beta\left(k+\frac{1}{k}\right)\frac{\pi}{2}\right]. \tag{12.69}$$

Hence, the solution of the modified Helmholtz equation in the interior of the equilateral triangle, with $l=\pi$ and with the Dirichlet boundary condition $d(s)=\cos(s)$ on each side of the triangle, is given by (71a), where D, G, Δ, Δ^+ are given by (12.68) and (12.69) and the rays l and l' are defined by (see Figure 8 of the introduction)

$$l = \left\{k\in\mathbb{C},\ \arg k = -\frac{\pi}{2}\right\},\quad l' = \left\{k\in\mathbb{C},\ \arg k = \phi,\ -\frac{\pi}{2}<\phi<-\frac{\pi}{6}\right\}.$$

Remark 12.5. It can be verified directly that the integrands of the integrals appearing in (71a) decay exponentially. Indeed, regarding the first integral, for which $\operatorname{Re} k = 0$, $\operatorname{Im} k < 0$, the following formulae are valid as $k\to 0$ or $k\to\infty$:

- $e^{i\beta\left(kz-\frac{\bar{z}}{k}\right)}E(-ik) \sim e^{\beta\left(ik+\frac{1}{ik}\right)\left(\operatorname{Re}(z)-\frac{\pi}{2\sqrt{3}}\right)} \sim e^{-\beta\left(t+\frac{1}{t}\right)\left(x-\frac{\pi}{2\sqrt{3}}\right)}$,
$$t<0,\quad x<\frac{\pi}{2\sqrt{3}};$$
- $D(k)\sim\dfrac{1}{k^2}$ as $k\to\infty$, $D(k)\sim k^2$ as $k\to 0$;
- $\dfrac{G(k)}{\Delta(k)} \sim e\left(-\dfrac{\sqrt{3}}{2}ik\right) \sim e^{\frac{\sqrt{3}}{2}\left(t+\frac{1}{t}\right)},\ t<0.$

For the second integral, for which $\arg k\in\left(-\frac{\pi}{2},-\frac{\pi}{6}\right)$, the following formulae are valid as $k\to 0$ or $k\to\infty$:

- $e^{i\beta(kz-\frac{1}{k})}E^2(iak) \sim \exp\left[\left(x-\frac{\pi}{2\sqrt{3}}\right)\cos\left(\phi+\frac{\pi}{2}\right)-\left(y+\frac{\pi}{2}\right)\sin\left(\phi+\frac{\pi}{2}\right)\right]$, where $x < \frac{\pi}{2\sqrt{3}}$, $y > -\frac{\pi}{2}$, and $\phi = \arg k$. Hence, since $\arg k \in \left(-\frac{\pi}{2}, -\frac{\pi}{6}\right)$, the argument of the exponential is negative;
- $\frac{G(k)}{\Delta(k)\Delta(ak)} \sim \frac{1}{k}$.

Similar considerations are valid for the other two terms of $A(k, z, \bar{z})$.

Remark 12.6. The Laplace, modified Helmholtz, and Helmholtz equations with the following oblique Robin boundary conditions are investigated in [32]:

$$\sin\beta_j q_n^{(j)}(s) + \cos\beta_j \frac{d}{ds}q^{(j)}(s) + \gamma_j q^{(j)}(s) = g_j(s),$$
$$s \in \left(-\frac{l}{2}, \frac{l}{2}\right), \quad j = 1, 2, 3, \tag{12.70}$$

where g_j are smooth functions and β_j, γ_j, $j = 1, 2, 3$, are real constants. It is shown that the unknown boundary values can be determined explicitly, provided that the following conditions are satisfied:

$$\begin{aligned} &\beta_2 = \beta_1 + \frac{m\pi}{3}, \quad \beta_3 = \beta_1 + \frac{n\pi}{3}, \quad m, n \in \mathbb{Z}, \\ &\sin 3\beta_1 \left[\gamma_2(3\beta^2 - \gamma_2^2) - e^{im\pi}\gamma_1(3\beta^2 - \gamma_1^2)\right] = 0, \\ &\sin 3\beta_1 \left[\gamma_3(3\beta^2 - \gamma_3^2) - e^{in\pi}\gamma_1(3\beta^2 - \gamma_1^2)\right] = 0. \end{aligned} \tag{12.71}$$

12.6 The Dirichlet to Neumann Correspondence

It was shown in section 1.4 that it is possible to determine the unknown Neumann boundary value $q_x(0, t)$ of the Dirichlet problem of the heat equation on the half-line *without* first determining $q(x, t)$. This was achieved by solving the global relation, which is formulated on the *boundary* of the relevant domain. Similar considerations are valid for evolution PDEs with derivatives of arbitrary order.

The situation with elliptic PDEs is similar: For problems that can be solved by the new transform method, it is possible to determine the unknown boundary values directly without determining $q(x, y)$. This will be illustrated below.

Example 12.4. Let $q(x, y)$ satisfy the Laplace equation in the quarter plane with the oblique Neumann boundary conditions described in Proposition 12.1. Let $u_2(x)$ denote the unknown derivative of $q(x, y)$ in the direction normal to the direction of the given boundary condition at $y = 0$; see (12.5b). The function $u_2(x)$ can be determined in terms of the given functions $g_1(y)$ and $g_2(x)$ by one of the following equations:

$$\int_0^\infty \sin(kx)u_2(x)dx = -\int_0^\infty \cos(kx)g_2(x)dx \pm \int_0^\infty e^{-ky}g_1(y)dy, \quad k \in \mathbb{R}^+, \tag{12.72$^\pm$}$$

or

$$\int_0^\infty \cos(kx)u_2(x)dx = \int_0^\infty \sin(kx)g_2(x)dx + \int_0^\infty e^{-ky}g_1(y)dy, \quad k \in \mathbb{R}^-, \qquad (12.73)$$

where $(12.72)^\pm$ correspond to $\beta_1 + \beta_2 = 0, \pi$, and (12.73) corresponds to $\beta_1 + \beta_2 = \pi/2$. The function $u_1(y)$, which denotes the unknown derivative of $q(x, y)$ in the direction normal to the direction of the given boundary condition at $x = 0$ (see (12.5a)), can be determined in a similar way.

Indeed, in this case, the global relation and its Schwarz conjugate are equations (12.8) which are both valid for $k = \mathbb{R}^-$. Eliminating the function $U_1(k)$ from these two equations, we find

$$i\left[U_2(ik) - e^{2i(\beta_1+\beta_2)}U_2(-ik)\right]$$
$$= G_2(ik) + e^{2i(\beta_1+\beta_2)}G_2(-ik) + 2e^{i(\beta_1+\beta_2)}G_1(k), \quad k \in \mathbb{R}^-.$$

Replacing k by $-k$ in this equation and letting $\beta_1 + \beta_2$ equal $0, \pi$, or $\pi/2$, we find equations $(12.72)^+$, $(12.72)^-$, or (12.73), respectively. In order to determine $u_1(y)$, we eliminate $U_2(-ik)$ from (12.8a) and from the equation obtained from (12.8b) by replacing k with $-k$.

Example 12.5. Let $q(x, y)$ satisfy the boundary value problem for the Laplace equation in the semi-infinite strip described in Proposition 12.2. The unknown Dirichlet boundary value $q(x, 0)$, $0 < x < \infty$, can be determined in terms of the given Neumann data $n(y)$, $0 < y < l$, and the Robin constant γ by

$$\int_0^\infty \cos(kx)q(x, 0)dx = \frac{1}{2}\frac{\int_0^l e^{ky}n(y)dy}{(k-\gamma) - e^{kl}(k+\gamma)}, \quad k \in \mathbb{R}. \qquad (12.74)$$

The unknown Dirichlet boundary value $q(0, y)$, $0 < y < l$, can be determined in a similar way.

Indeed, in this case, the global relation and its Schwarz conjugate are equations (12.23) which are both valid for $k = \mathbb{R}$. Subtracting these equations we find (12.74). The easiest way to find $q(0, y)$ is to use both equations (12.23) (with U_3 already known) and the inverse Fourier transform of $U_2(k)$.

Example 12.6. Let $q(x, y)$ satisfy the Dirichlet problem for the modified Helmholtz equation in the semi-infinite strip described in Proposition 12.3. The unknown Neumann boundary values can be determined by the following equations:

$$2i\int_0^\infty \sin\left[\beta\left(k - \frac{1}{k}\right)x\right]q_y(x, 0)dx = \beta F_3(k), \quad k \in \mathbb{R}^+, \qquad (12.75a)$$

$$2i\int_0^\infty \sin\left[\beta\left(k - \frac{1}{k}\right)x\right]q_y(x, l)dx = -\beta F_1(k), \quad k \in \mathbb{R}^+, \qquad (12.75b)$$

$$-2i\int_0^l \sin\left(\frac{n\pi}{l}y\right)q_x(0, y)dy = \overline{J(\bar{k}_n)} - J(-k_n),$$

$$k_n = \frac{i}{2}\left[\frac{\pi n}{\beta l} + \sqrt{\frac{\pi^2 n^2}{\beta^2 l^2} + 4}\right], \quad n \in \mathbb{Z}^+, \tag{12.75c}$$

where the functions F_1, F_3, J are defined in terms of the given Dirichlet boundary conditions by (12.31e), (12.31f), (12.34b), respectively.

Indeed, in this case, by manipulating the global relation and its Schwarz conjugate we find equations (12.36), which are (12.75a) and (12.75b). In order to determine $q_x(0, y)$, $0 < y < l$, we subtract (12.34c) from the equation obtained from (12.34a) by replacing k with $-k$:

$$[E(k) - E(-k)]U_1(ik) + U_2(k) - U_2(-k) = \overline{J(\bar{k})} - J(-k), \quad \operatorname{Im} k \geq 0.$$

In order to eliminate $U_1(ik)$ we evaluate this equation at those values of k in $\mathbb{C}^+$ for which the coefficient of $U_1(ik)$ vanishes, i.e., at $k = k_n$, where

$$e^{\beta\left(k_n + \frac{1}{k_n}\right)l} = e^{-\beta\left(k_n + \frac{1}{k_n}\right)l}, \quad k_n \in \mathbb{C}^+; \quad e^{\beta l\left(k_n + \frac{1}{k_n}\right)} = e^{i\pi n}, \quad n \in \mathbb{Z}.$$

This evaluation, using the definition of $U_2(k)$ (see (12.33b)), implies (12.75c).

Example 12.7. Let $q(x, y)$ satisfy the symmetric Dirichlet problem for the modified Helmholtz equation in the equilateral triangle described in Proposition 12.5. The unknown Neumann boundary value $q_n(s)$ on each side of the triangle can be determined from the equation

$$2\sinh\left[\frac{\beta l}{2}\left(\alpha k_n + \frac{1}{\alpha k_n}\right)\right]\int_{-\frac{l}{2}}^{\frac{l}{2}} e^{\frac{2i\pi n}{l}s} q_n(s)ds = -i\beta\left[J(k_n) + e^{i\pi n}\overline{J(\bar{k}_n)}\right], \tag{12.76}$$

$$k_n = i\left[\frac{\pi n}{\beta l} + \sqrt{\frac{\pi^2 n^2}{\beta^2 l^2} + 1}\right], \quad n \in \mathbb{Z},$$

where the function $J(k)$ is defined in terms of the given Dirichlet boundary condition $d(s)$ by (12.55).

Indeed, in this case, by manipulating the global relation and its Schwarz conjugate we find (12.58). This equation is a single equation for the two unknown functions $U(k)$ and $U(\alpha k)$. However, by evaluating this equation at those values of k for which the coefficient of $U(\alpha k)$ vanishes, i.e., at $k = k_n$, where

$$e^{\frac{\beta l}{2}\left(k_n + \frac{1}{k_n}\right)} = e^{-\frac{\beta l}{2}\left(k_n + \frac{1}{k_n}\right)}, \quad \text{or} \quad \beta l\left(k_n + \frac{1}{k_n}\right) = 2i\pi n, \quad n \in \mathbb{Z},$$

(12.58) yields (12.76) (recall that $U(k)$ is defined by (68) from the introduction).

Example 12.8. Let $q(x, y)$ satisfy the symmetric Dirichlet problem for the equilateral triangle described in Proposition 12.5, but for the Helmholtz equation (9.26) instead of the modified Helmholtz equation. The unknown Neumann boundary value $q_n(s)$ on each side of the triangle can be determined from the equation

$$2\sinh\left[\frac{\beta l}{2}\left(\bar{\alpha}k_n - \frac{1}{\bar{\alpha}k_n}\right)\right]\int_{-\frac{l}{2}}^{\frac{l}{2}} e^{\frac{2i\pi n}{l}s} q_n(s)ds = i\beta\left[J(k_n) + e^{i\pi n}\overline{J(\bar{k}_n)}\right],$$

$$k_n = i\left[\frac{\pi n}{\beta l} + \sqrt{\frac{\pi^2 n^2}{\beta^2 l^2} - 1}\right], \quad n \in \mathbb{Z}, \tag{12.77}$$

where the function $J(k)$, $k \in \mathbb{C}$, is defined in terms of the given Dirichlet boundary condition $d(s)$ by

$$J(k) = \left(k + \frac{1}{k}\right) e(-\alpha k)D(k) + e(k)\left(\alpha k + \frac{1}{\alpha k}\right) D(\alpha k) + \left(\bar{\alpha}k + \frac{1}{\bar{\alpha}k}\right) D(\bar{\alpha}k),$$

$$e(k) = e^{\frac{\beta l}{2}\left(k - \frac{1}{k}\right)}, \quad D(k) = \int_{-\frac{l}{2}}^{\frac{l}{2}} e^{\beta\left(k - \frac{1}{k}\right)s} d(s)ds, \tag{12.78}$$

where $\overline{J(\bar{k})}$ denotes the function obtained from $J(k)$ by taking the complex conjugate of all the terms in $J(k)$ except $d(s)$.

Indeed, in this case, the first of the global relations is given by (see (9.30))

$$\hat{q}(k) + \hat{q}(\alpha k) + \hat{q}(\bar{\alpha}k), \quad k \in \mathbb{C},$$

where

$$\hat{q}(k) = \int_{-\frac{l}{2}}^{\frac{l}{2}} e^{-i\beta\left(kz + \frac{\bar{z}}{k}\right)}\left[q_n + \beta\left(k\frac{dz}{ds} - \frac{1}{k}\frac{d\bar{z}}{ds}\right) q\right] ds, \quad k \in \mathbb{C}.$$

Using the parametrizations defined by (11.23), it follows that

$$\hat{q}(k) = E(-ik)\left[U(k) + i\beta\left(k + \frac{1}{k}\right) D(k)\right],$$

where $D(k)$ is as defined in (12.78), whereas $E(k)$ and $U(k)$ are defined by

$$E(k) = e^{\frac{l}{2\sqrt{3}}\left(k - \frac{1}{k}\right)}, \quad U(k) = \int_{-\frac{l}{2}}^{\frac{l}{2}} e^{\beta\left(k - \frac{1}{k}\right)s} q_n(s)ds. \tag{12.79}$$

Substituting the representations of $\hat{q}(k), \hat{q}(\alpha k), \hat{q}(\bar{\alpha}k)$ in the global relation and multiplying the resulting equation by $E(i\alpha k)$, in analogy with (12.54), we now find

$$e(-\alpha k)U(k) + e(k)U(\alpha k) + U(\bar{\alpha}k) = -i\beta J(k),$$

where $e(k)$ and $J(k)$ are as defined in (12.78). Following precisely the same steps used in Example 12.71, we find (12.77).

Remark 12.7. Comparing the derivation of the *solution* $q(x, y)$ presented in sections 12.1–12.5 with the derivation of the unknown *boundary values* presented in section 12.6, it follows that the latter derivation is much simpler. However, the expressions for the boundary values obtained by evaluating the solution $q(x, y)$ on the boundary have the advantage that they involve integrals with exponential decay. This has important analytical and numerical implications.

Remark 12.8. It was shown in Part I that it is possible to obtain the novel representation for $q(x, t)$ by analyzing the global relation in the *subdomain* $0 < x < \infty$, $0 < s < t$. Similarly, the novel representations for elliptic PDEs can be obtained by analyzing the global relations in certain appropriate subdomains [24]. Actually, the algebraic manipulation of these representations also provides an alternative approach to classical transforms for deriving the classical representations of $q(x, y)$.

Chapter 13

Formulation of Riemann–Hilbert Problems

The novel integral representations derived in Chapter 11 express the solution $q(x, y)$ of the basic elliptic PDEs in terms of certain transforms of the boundary values denoted by $\{\hat{q}_j(k)\}_1^n$. These functions are coupled by the two global relations. For the case of simple polygonal domains, the algebraic manipulation of the global relations and of the equations obtained from the global relations via certain transformations in the complex k-plane immediately yield a Riemann–Hilbert (RH) problem for the characterization of $\{\hat{q}_j(k)\}^n$. In what follows we illustrate this approach for the simple cases of the quarter plane and the semi-infinite strip.

13.1 The Laplace Equation in the Quarter Plane

Laplace's equation in the quarter plane with the oblique Robin boundary conditions (12.14) is analyzed in [29]. Here, for simplicity we consider only the case of oblique Neumann boundary conditions.

Proposition 13.1. Let the complex-valued function $q(x, y)$ satisfy the oblique Neumann problem for the Laplace equation in the quarter plane described in Proposition 12.1, but *without* the restriction (12.2b) on the constants β_1 and β_2. Let $u_1(y)$ and $u_2(x)$ denote the unknown derivatives of the given boundary conditions; see (12.5). Let $U_1(k)$ and $U_2(k)$ denote the transforms of $u_1(y)$ and $u_2(x)$ defined by (12.7). The function $U_2(k)$ satisfies the following scalar RH problem:

$$U_2(ik) \ \text{is analytic for} \ \operatorname{Im} k > 0, \tag{13.1a}$$

$$U_2(ik) - J(k)U_2(-ik) = \chi(k), \quad k \in \mathbb{R}, \tag{13.1b}$$

$$U_2(k) = o(1), \quad k \to \infty, \tag{13.1c}$$

where

$$J(k) = \begin{cases} e^{2i(\beta_1+\beta_2)}, & k < 0, \\ e^{-2i(\beta_1+\beta_2)}, & k > 0, \end{cases}$$

$$\chi(k) = \begin{cases} -2ie^{i(\beta_1+\beta_2)}G_1(k) - ie^{2i(\beta_1+\beta_2)}G_2(-ik) - iG_2(ik), & k < 0, \\ 2ie^{-i(\beta_1+\beta_2)}G_1(-k) + iG_2(ik) + ie^{-2i(\beta_1+\beta_2)}G_2(-ik), & k > 0, \end{cases} \tag{13.2}$$

and the functions $G_1(k)$, $G_2(k)$ are defined in terms of the given boundary conditions $g_1(k)$, $g_2(k)$ by (12.3).

The function $U_1(k)$ satisfies a similar scalar RH problem.

Proof. In this case the two global relations are equations (12.8), which are both valid for $k = \mathbb{R}^-$. Eliminating from these equations the function $U_1(k)$ we find (13.1b) for $k < 0$. Replacing k with $-k$ in this equation we find (13.1b) for $k > 0$.

The definition of $U_2(k)$ implies (13.1a) and (13.1c).

The associated RH problem for $U_1(k)$ can be obtained in a similar way by eliminating $U_2(ik)$ from (12.8b) and from the equation obtained from (12.8a) by replacing k with $-k$. □

Remark 13.1. The scalar RH problem (13.1) is discontinuous at $k = 0$ unless β_1 and β_2 satisfy condition (12.2b). The RH problem (13.1) can be solved in a closed form by standard methods; see, for example, [17], [86], [93].

13.2 The Laplace Equation in a Semi-Infinite Strip

Laplace's equation in a semi-infinite strip with the oblique Robin boundary conditions (12.26) is analyzed in [29]. Here, for simplicity we consider only the case of oblique Neumann boundary conditions.

Proposition 13.2. Let the complex-valued function $q(x, y)$ satisfy the Laplace equation in the semi-infinite strip $\{0 < x < \infty, 0 < y < l\}$ with the following oblique Neumann boundary conditions (see Figure 12.5):

$$\begin{aligned} &q_x \cos\beta_1 + q_y \sin\beta_1 = g_1(x), \quad 0 < x < \infty, \quad y = l, \\ &q_y \cos\beta_2 - q_x \sin\beta_2 = g_2(y), \quad x = 0, \quad 0 < y < l, \\ &q_x \cos\beta_3 - q_y \sin\beta_3 = g_3(x), \quad 0 < x < \infty, \quad y = 0, \end{aligned} \tag{13.3}$$

where the complex-valued functions $\{g_j\}_1^3$ have sufficient smoothness, the functions g_1, g_3 have sufficient decay, and $\{\beta_j\}_1^3$ are real constants. Let $\{u_j\}_1^3$ denote the derivatives in the directions normal to the directions of the given deviations, i.e.,

$$\begin{aligned} &q_x \sin\beta_1 - q_y \cos\beta_1 = u_1(x), \quad 0 < x < \infty, \quad y = l, \\ &-q_y \sin\beta_2 - q_x \cos\beta_2 = u_2(y), \quad x = 0, \quad 0 < y < l, \\ &q_x \sin\beta_3 + q_y \cos\beta_3 = u_3(x), \quad 0 < x < \infty, \quad y = 0. \end{aligned} \tag{13.4}$$

Let G_j and U_j, $j = 1, 2, 3$, denote the following transforms of g_j and u_j, respectively:

$$G_1(k) = -\frac{1}{2}\int_0^\infty e^{kx} g_1(x)dx, \quad G_2(k) = -\frac{1}{2}\int_0^l e^{ky} g_2(y)dy,$$

$$G_3(k) = \frac{1}{2}\int_0^\infty e^{kx} g_3(x)dx, \tag{13.5}$$

$$U_1(k) = -\frac{1}{2}\int_0^\infty e^{kx} u_1(x)dx, \quad U_2(k) = \frac{1}{2}\int_0^l e^{ky} u_2(y)dy,$$

$$U_3(k) = -\frac{1}{2}\int_0^\infty e^{kx} u_3(x)dx, \tag{13.6}$$

where G_2, U_2 are defined for all $k \in \mathbb{C}$, whereas G_1, G_3, U_1, U_3 are defined for $\operatorname{Re} k \leq 0$.

The functions $\{U_1, U_3\}$ satisfy the following 2×2 matrix RH problem:

$$U_1(ik), \quad U_3(ik) \quad \text{are analytic for} \quad \operatorname{Im} k > 0, \tag{13.7a}$$

$$J(k)\begin{pmatrix} U_1(ik) \\ U_3(ik) \end{pmatrix} - \bar{J}(k)\begin{pmatrix} U_1(-ik) \\ U_3(-ik) \end{pmatrix} = \begin{pmatrix} -i\chi(k) \\ i\chi(-k) \end{pmatrix}, \quad k \in \mathbb{R}, \tag{13.7b}$$

$$U_1(ik) = o(1), \quad U_3(ik) = o(1), \quad k \to \infty, \tag{13.7c}$$

where $J(k)$ and $\chi(k)$ are defined by

$$J(k) = \begin{pmatrix} e^{i(\beta_1-\beta_2)}e^{kl} & e^{-i(\beta_2+\beta_3)} \\ e^{i(\beta_2-\beta_1)}e^{-kl} & e^{i(\beta_2+\beta_3)} \end{pmatrix}, \quad k \in \mathbb{C}, \tag{13.8a}$$

$$\begin{aligned}\chi(k) = 2G_2(k) &+ e^{kl}\left[e^{i(\beta_2-\beta_1)}G_1(-ik) + e^{i(\beta_1-\beta_2)}G_1(ik)\right] \\ &+ e^{i(\beta_2+\beta_3)}G_3(-ik) + e^{-i(\beta_2+\beta_3)}G_3(ik), \quad k \in \mathbb{R}.\end{aligned} \tag{13.8b}$$

Proof. The boundary conditions of the sides (2) and (3) are the same as the boundary conditions on the sides (1) and (2) of the quarter plane. Thus, we define the unknown functions u_2 and u_3 by (13.4b) and (13.4c), which can be obtained from (12.5) by the substitutions $1 \to 2$ and $2 \to 3$. Then we obtain for $\hat{q}_2$ and $\hat{q}_3$ the equations obtained from (12.6) with $1 \to 2$ and $2 \to 3$, i.e.,

$$\hat{q}_2(k) = e^{-i\beta_2}[G_2(k) + iU_2(k)], \quad \hat{q}_3(k) = e^{i\beta_3}[G_3(-ik) + iU_3(-ik)], \tag{13.9}$$

where U_2, U_3, G_2, G_3 are defined in (13.5) and (13.6); these equations follow from (12.3) and (12.7) using $1 \to 2$ and $2 \to 3$.

Solving (13.3a) and (13.4a) for q_x and q_y we find

$$q_x(x,l) = g_1(x)\cos\beta_1 + u_1(x)\sin\beta_1, \quad q_y(x,l) = g_1(x)\sin\beta_1 - u_1(x)\cos\beta_1.$$

Substituting these expressions in the equation

$$\hat{q}_1(k) = -\frac{1}{2}\int_0^\infty e^{-ik(x+il)}[q_x(x,l) - iq_y(x,l)]dx,$$

we find

$$\hat{q}_1(k) = e^{kl}e^{-i\beta_1}[G_1(-ik) + iU_1(-ik)], \quad \operatorname{Im} k \le 0. \tag{13.10}$$

Substituting the expressions for $\{\hat{q}_j\}_1^3$ from (13.9) and (13.10) into the global relation, we find

$$\begin{aligned} e^{-i\beta_1}e^{-kl}[G_1(-ik) + iU_1(-ik)] + e^{-i\beta_2}[G_2(k) + iU_2(k)] \\ + e^{i\beta_3}[G_3(-ik) + iU_3(-ik)] = 0, \quad \operatorname{Im} k \le 0. \end{aligned} \tag{13.11}$$

Taking the Schwarz conjugate of this equation and then eliminating $U_2(k)$ from the resulting equation and from (13.11), we find the following equation valid for $k \in \mathbb{R}$:

$$\begin{aligned} &\left[e^{i(\beta_1-\beta_2)}e^{kl}U_1(ik) + e^{-i(\beta_2+\beta_3)}U_3(ik)\right] \\ &\quad - \left[e^{i(\beta_2-\beta_1)}e^{kl}U_1(-ik) + e^{i(\beta_2+\beta_3)}U_3(-ik)\right] = -i\chi(k). \end{aligned} \tag{13.12}$$

Replacing k by $-k$ in (13.12) and writing the resulting equation and (13.12) in matrix form we find (13.7b).

The definitions of $U_1(ik)$ and $U_3(ik)$ imply (13.7a) and (13.7c). □

Remark 13.2. The (2-1) and (1-2) entries of the matrix $J(k)^{-1}\overline{J(\bar{k})}$ are proportional to

$$e^{2i(\beta_1-\beta_2)} - e^{2i(\beta_2-\beta_1)}, \quad e^{2i(\beta_2+\beta_3)} - e^{-2i(\beta_2+\beta_3)}.$$

Hence, if *either* of the following conditions is valid, i.e.,

$$e^{4i(\beta_1-\beta_2)} = 1 \text{ or } e^{4i(\beta_2+\beta_3)} = 1,$$

the RH problem defined by (13.7) becomes triangular and hence is reduced to a scalar RH problem that can be solved in closed form. If *both* the above conditions are satisfied, then the above RH problem can be bypassed; see Remark 12.2. Similarly, if either of equations (12.28) is valid, then the corresponding RH problem associated with the oblique Robin boundary conditions (12.26) is triangular [29].

13.3 The Modified Helmholtz Equation in a Semi-Infinite Strip

Proposition 13.3. Let the complex-valued function $q(x, y)$ satisfy the modified Helmholtz equation in the semi-infinite strip $\{0 < x < \infty, 0 < y < l\}$, with the oblique Robin boundary conditions (12.26)—see Figure 12.5—where the complex-valued functions $\{g_j\}_1^3$ have sufficient smoothness, the functions g_1, g_3 have sufficient decay, and β_j, γ_j are real constants with $\sin\beta_j \neq 0$, $j = 1, 2, 3$. Let $\{U_j(k)\}_1^3$ denote the following transforms of the unknown Dirichlet boundary values:

$$U_1(k) = \frac{1}{\sin\beta_1}\int_0^\infty e^{\beta(k+\frac{1}{k})x}q(x,l)dx, \quad U_3(k) = \frac{1}{\sin\beta_3}\int_0^\infty e^{\beta(k+\frac{1}{k})x}q(x,0)dx,$$

$$U_2(k) = \frac{1}{\sin\beta_2}\int_0^l e^{\beta(k+\frac{1}{k})y} q(0,y)dy, \tag{13.13}$$

where $U_1(k)$, $U_3(k)$ are defined for $\mathrm{Re}\,k \le 0$, whereas $U_2(k)$ is defined for all $k \in \mathbb{C}$.

Let $\{G_j\}_1^3$ denote the following transforms of the given boundary conditions:

$$G_j(k) = \frac{1}{\sin\beta_j}\int_0^\infty e^{\beta(k+\frac{1}{k})x} g_j(x)dx, \quad j = 1,3, \quad \mathrm{Re}\,k \le 0,$$

$$G_2(k) = \frac{1}{\sin\beta_2}\int_0^l e^{\beta(k+\frac{1}{k})x} g_2(y)dy, \quad k \in \mathbb{C}. \tag{13.14}$$

The functions U_1 and U_3 satisfy the following 2×2 matrix RH problem:

$$U_1(ik), \;\; U_3(ik) \;\; \text{are analytic for} \;\; \mathrm{Im}\,k > 0, \tag{13.15a}$$

$$J(k)\begin{pmatrix} U_1(ik) \\ U_3(ik) \end{pmatrix} + \overline{J(\bar{k})}\begin{pmatrix} U_1(-ik) \\ U_3(-ik) \end{pmatrix} = \begin{pmatrix} \chi(k) \\ \chi(-k) \end{pmatrix}, \quad k \in \mathbb{R}, \tag{13.15b}$$

$$U_1(ik) = o(1), \;\; U_3(ik) = o(1), \;\; k \to \infty \;\; \text{and} \;\; k \to 0, \tag{13.15c}$$

where $J_1(k)$ and $\chi(k)$ are defined by the following equations:

$$J(k) = \begin{pmatrix} E(ik)\overline{J_1(\bar{k})}J_2(k) & J_2(k)\overline{J_3(\bar{k})} \\ E(ik)J_1(-k)\overline{J_2(-\bar{k})} & \overline{J_2(-\bar{k})}J_3(-k) \end{pmatrix}, \quad k \in \mathbb{R}, \tag{13.16}$$

$$E(k) = e^{\beta l(k+\frac{1}{k})}, \quad J_1(k) = \beta\left(e^{-i\beta_1}k - \frac{e^{i\beta_1}}{k}\right) - i\gamma_1,$$

$$J_2(k) = \beta\left(e^{-i\beta_2}k + \frac{e^{i\beta_2}}{k}\right) - \gamma_2, \;\; J_3(k) = \beta\left(e^{i\beta_3}k - \frac{e^{-i\beta_3}}{k}\right) - i\gamma_3, \;\; k \in \mathbb{C}, \tag{13.17}$$

$$\begin{aligned}\chi(k) &= iE(ik)J_2(k)G_1(ik) - iE(-ik)\overline{J_2(\bar{k})}G_1(-ik) + 2\sin\beta_2\left(k - \frac{1}{k}\right)G_2(k) \\ &+ i\left[J_2(k)G_3(ik) - \overline{J_2(\bar{k})}G_3(ik)\right] + i\delta_1\cot\beta_1\left[E(ik)J_2(k) - E(-ik)\overline{J_2(\bar{k})}\right] \\ &+ 2\beta\left(k - \frac{1}{k}\right)\left[\delta_2(\cos\beta_2 + \sin\beta_2\cot\beta_3) - \delta_1\cos\beta_2 E(k)\right],\end{aligned}$$

$$\delta_1 = q(0,l), \quad \delta_2 = q(0,0). \tag{13.18}$$

Proof. In order to formulate the global relation we must first use the boundary conditions to express the functions $\{\hat{q}_j\}_1^3$ defined by (11.40) in terms of $\hat{g}_j$ and U_j, $j = 1, 2, 3$. The function $\hat{q}_1$ involves $q(x,l)$ and $q_y(x,l)$. Solving (12.26a) for q_y, substituting the resulting expression in (11.40a), and using integration by parts to eliminate $q_x(x,l)$, we find

$$\hat{q}_1(k) = E(-ik)\left[iG_1(-ik) + J_1(k)U_1(-ik) + i\cot\beta_1 q(0,l)\right], \;\; \mathrm{Im}\,k \le 0. \tag{13.19a}$$

Similarly, solving (12.26b) and (12.26c) for $q_x(0,y)$ and $g_y(x,0)$ and substituting the resulting expression in (11.40b) and (11.40c), we find

$$\hat{q}_2(k) = iG_2(k) + iJ_2(k)U_2(k) + i\cot\beta_2[q(0,0) - E(k)q(0,l)], \quad k \in \mathbb{C}, \tag{13.19b}$$

$$\hat{q}_3(k) = iG_3(-ik) + J_3(k)U_3(-ik) + i\cot\beta_3 q(0,0), \quad \operatorname{Im} k \le 0. \tag{13.19c}$$

Substituting the expressions for $\{\hat{q}_j\}_1^3$ from (13.19) into the global relation, taking the Schwarz conjugate of the resulting equation, and then eliminating the function $U_2(k)$ from these two equations, we find the following equation which is valid for $k \in \mathbb{R}$:

$$\begin{aligned} &\left[E(ik)\overline{J_1(\bar{k})}J_2(k)U_1(ik) + J_2(k)\overline{J_3(\bar{k})}U_3(ik)\right] \\ &\quad + \left[E(-ik)J_1(k)\overline{J_2(\bar{k})}U_1(-ik) + \overline{J_2(\bar{k})}J_3(k)U_3(-ik)\right] = \chi(k). \end{aligned} \tag{13.20}$$

Replacing k with $-k$ in this equation and writing the resulting equation and (13.20) in matrix form; we find (13.15b).

The definitions of $U_1(ik)$ and $U_3(ik)$ imply (13.15a) and (13.15c). □

Remark 13.3. The solvability of the RH problem (13.5), as well as the question of how to determine the values $q(0,0)$ and $q(0,l)$, is discussed in [30]. Here we only note that if *either* the conditions

$$e^{4i(\beta_2-\beta_1)} = 1 \ \text{ and } \ (2\beta^2 - \gamma_2^2)\sin 2\beta_1 + (2\beta^2 - \gamma_1^2)\sin 2\beta_2 = 0 \tag{13.21}$$

or

$$e^{4i(\beta_2+\beta_3)} = 1 \ \text{ and } \ (2\beta^2 - \gamma_2^2)\sin 2\beta_3 - (2\beta^2 - \gamma_3^2)\sin 2\beta_2 = 0 \tag{13.22}$$

are satisfied, then the RH problem defined by (13.15) becomes triangular and hence is reduced to a scalar RH problem that can be solved in closed form. Indeed, the (1-2) entry of the relevant jump matrix is proportional to $A(k) - \bar{A}(k)$, where

$$A(k) = [J_3(k)J_3(-k)]\bar{J}_2(k)\bar{J}_2(-k).$$

Consider for brevity of presentation the case that $\gamma_2 = \gamma_3 = 0$. Then

$$J_3(k)J_3(-k) = -\beta^2\left(e^{i\beta_3}k - \frac{e^{-i\beta_3}}{k}\right)^2, \quad \bar{J}_2(k)\bar{J}_2(-k) = -\beta^2\left(e^{i\beta_2}k + \frac{e^{-i\beta_2}}{k}\right)^2,$$

$$\begin{aligned} A(k) - \bar{A}(k) &= \beta^4\left(k^4 - \frac{1}{k^4}\right)\left[e^{2i(\beta_2+\beta_3)} - e^{-2i(\beta_2+\beta_3)}\right] \\ &\quad + 2\beta^2\left(k^2 - \frac{1}{k^2}\right)\left[\left(e^{-2i\beta_2} - e^{2i\beta_2}\right) + \left(e^{2i\beta_3} - e^{-2i\beta_3}\right)\right]. \end{aligned} \tag{13.23}$$

The coefficient of $k^4 - 1/k^4$ vanishes if and only if the first of equations (13.22) is valid; the coefficient of $k^2 - 1/k^2$ vanishes as a consequence of the second of equations (13.22) (here $\gamma_2 = \gamma_3 = 0$).

Chapter 14

A Collocation Method in the Fourier Plane

It was shown in Proposition 1 of the introduction that the transforms $\hat{q}_j(k)$ and $\tilde{q}_j(k)$ of the boundary values of the solution of the modified Helmholtz equation in the interior of a convex polygon are coupled by the *two* global relations (52). Similar relations are valid for the Laplace and the Helmholtz equation. For simple boundary value problems the algebraic manipulation of the global relations yields the unknown boundary values through the inversion of elementary integrals; see section 12.6. For more complicated boundary value problems, the algebraic manipulation of the global relations characterizes the unknown boundary values through the solution of scalar or matrix Riemann–Hilbert (RH) problems; see Chapter 13. In what follows we present a simple technique for the numerical evaluation of the unknown boundary values.

14.1 The Laplace Equation

For simplicity we consider the case of oblique Neumann boundary conditions; other boundary value problems can be treated in a similar manner.

Proposition 14.1. Let the complex-valued function $q(z, \bar{z})$ satisfy the Laplace equation in the interior of a convex polygon with corners $\{z_j\}_1^n$ (indexed counterclockwise, modulo n); see Figure 6 of the introduction. Let S_j denote the side (z_j, z_{j+1}). Suppose that the derivative of q making an angle β_j with the side S_j is prescribed on each side, namely

$$\cos\beta_j \frac{d}{ds} q_j(s) + \sin\beta_j q_j^n(s) = g_j(s), \quad j = 1, \ldots, n, \tag{14.1}$$

where s parametrizes the side S_j, q_j and q_j^n denote q and the derivative of q in the direction of the outward normal to the side S_j, each of the real constants $\{\beta_j\}_1^n$ satisfies $0 \le \beta_j \le \pi$, and the given complex-valued function g_j has sufficient smoothness. Let $u_j(s)$ denote the unknown derivative of q in the direction normal to the direction of the prescribed derivative, i.e.,

$$u_j(s) = -\sin\beta_j \frac{d}{ds} q_j(s) + \cos\beta_j q_j^n(s), \quad j = 1, \ldots, n. \tag{14.2}$$

The n unknown complex-valued functions $\{u_j\}_1^n$ satisfy the following $2n$ equations for all $l \in \mathbb{R}^+$ and $p = 1, \ldots, n$:

$$\int_{-\pi}^{\pi} e^{ils} u_p(s)ds = -\sum_{j=1}^{n} E_{jp}(l) \int_{-\pi}^{\pi} e^{il\frac{h_j}{h_p}s} u_j(s)ds + iG_p(l), \tag{14.3a}$$

$$\int_{-\pi}^{\pi} e^{-ils} u_p(s)ds = -\sum_{\substack{j=1 \\ j\neq p}}^{n} \bar{E}_{jp}(l) \int_{-\pi}^{\pi} e^{-il\frac{\bar{h}_j}{\bar{h}_p}s} u_j(s)ds - i\tilde{G}_p(l), \tag{14.3b}$$

where the known functions $E_{jp}(l)$, $G_p(l)$, and $\tilde{G}_p(l)$ are defined by

$$E_{jp}(l) = \exp\left[i(\beta_j - \beta_p) + \frac{il}{h_p}(m_j - m_p)\right], \tag{14.4a}$$

$$G_p(l) = \sum_{j=1}^{n} E_{jp}(l) \int_{-\pi}^{\pi} e^{il\frac{h_j}{h_p}s} g_j(s)ds, \tag{14.4b}$$

$$\tilde{G}_p(l) = \sum_{j=1}^{n} \bar{E}_{jp}(l) \int_{-\pi}^{\pi} e^{-il\frac{\bar{h}_j}{\bar{h}_p}s} g_j(s)ds, \tag{14.4c}$$

with

$$h_j = \frac{1}{2\pi}\left(z_{j+1} - z_j\right), \quad m_j = \frac{1}{2}\left(z_{j+1} + z_j\right), \quad j = 1, \ldots, n. \tag{14.5}$$

Furthermore, each of the terms appearing in the summations on the RHS of (14.3) decays exponentially as $l \to \infty$, except the terms with $j = p$ in G_p and $\tilde{G}_p$ which oscillate, as well as the terms with $j = p - 1$ and $j = p + 1$ which decay linearly.

Proof. Solving the two algebraic equations (14.1) and (14.2) for $q_j^s = dq_j/ds$ and q_j^n, we find

$$q_j^s = \cos\beta_j g_j - \sin\beta_j u_j,$$

$$q_j^n = \sin\beta_j g_j + \cos\beta_j u_j, \quad j = 1, \ldots, n.$$

Substituting these expressions in the identity

$$q_z dz = \frac{1}{2}\left[q_j^s(s) + iq_j^n(s)\right]ds, \tag{14.6}$$

we find for the side S_j,

$$q_z dz = \frac{1}{2}e^{i\beta_j}\left[g_j(s) + iu_j(s)\right]ds. \tag{14.7}$$

We parametrize the side S_j with respect to its midpoint m_j, i.e.,

$$z(s) = m_j + sh_j, \quad -\pi < s < \pi. \tag{14.8}$$

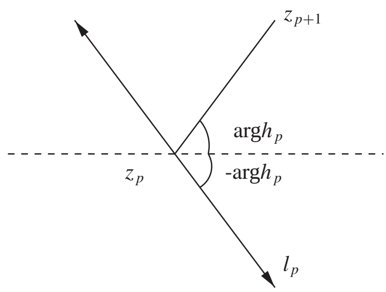


Figure 14.1. *The ray l_p.*

Substituting the expressions (14.7) and (14.8) in the definition of $\hat{q}_j(k)$, i.e., in the equation

$$\hat{q}_j(k) = \int_{z_j}^{z_{j+1}} e^{-ikz} q_z dz,$$

we find

$$\hat{q}_j(k) = \frac{1}{2} e^{i\beta_j - ikm_j} \int_{-\pi}^{\pi} e^{-ikh_j s} \left[g_j(s) + iu_j(s)\right] ds, \quad j = 1, \ldots, n. \tag{14.9}$$

Writing the first global relation, i.e., equation (11.6a), in the form

$$\hat{q}_p(k) = -\sum_{\substack{j=1 \\ j \neq p}}^{n} \hat{q}_j(k),$$

replacing $\hat{q}_j$ by the RHS of (14.9) in this equation and evaluating the resulting equation at

$$k = -\frac{l}{h_p}, \quad l \in \mathbb{R}^+, \tag{14.10}$$

we find (14.3a).

The second global relation, i.e., equation (11.6b), can be obtained from the first global relation by taking the Schwarz conjugate of all terms in (11.6a) except q. Thus, by taking the Schwarz conjugate of all terms in (14.3a) except u_j and g_j we find (14.3b).

Regarding the choice of k specified by (14.10), we recall that the ray l_p appearing in the integral representation of q_z (see (11.3)), is defined by

$$l_p = \{\arg k = -\arg\left(z_{p+1} - z_p\right) = -\arg h_p\}.$$

However, the integral representation involves $\exp[ikz]$, whereas the functions $\{\hat{q}_j\}_1^n$ involve $\exp[-ikz]$; thus associated with $\hat{q}_p$ we require that k satisfies the condition $\arg k = \pi - \arg(h_p)$, which is consistent with (14.10), since this latter equation implies (see Figure 14.1)

$$k = -\frac{l}{|h_p|} e^{-i\arg(h_p)} = \frac{l}{|h_p|} e^{i[\pi - \arg(h_p)]}, \quad \frac{l}{|h_p|} > 0.$$

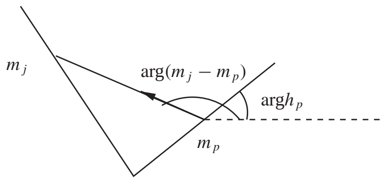


Figure 14.2. *The sides S_j and S_p.*

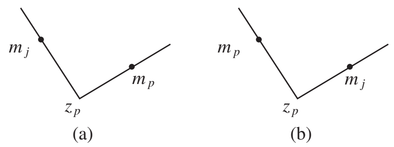


Figure 14.3. *The sides S_j and S_p.*

Convexity implies the estimate

$$\left[\arg(m_j - m_p) - \arg(h_p)\right] \in (0, \pi), \quad j \neq p. \tag{14.11}$$

Indeed, the above bracket equals the interior angle made by the sides S_j and S_p, see Figure 14.2.

Each of the terms appearing in the RHS of (14.3a) involves

$$\int_{-\pi}^{\pi} e^{\frac{il}{h_p}(m_j + sh_j - m_p)} a_j(s)ds, \quad a_j = u_j \text{ or } a_j = g_j. \tag{14.12}$$

Convexity also implies that for $s \in (0, \pi)$, the following inequality is valid:

$$j \neq p, \quad j \neq p \pm 1: \quad 0 < \left[\arg(m_j + sh_j - m_p) - \arg h_p\right] < \pi.$$

However, if $j = p - 1$, then (see Figure 14.3(a))

$$z_p = m_j + \pi h_j = m_p - \pi h_p,$$

hence,

$$j = p - 1: \quad \arg(m_j + \pi h_j - m_p) - \arg h_p = \pi.$$

Similarly, if $j = p + 1$, then (see Figure 14.3(b))

$$z_{p+1} = m_p + \pi h_p = m_j - \pi h_j,$$

hence

$$j = p + 1: \quad \arg(m_j - \pi h_j - m_p) - \arg h_p = 0.$$

Integration by parts implies that the LHS of (14.12) equals

$$\frac{h_p}{ilh_j}\left[e^{\frac{il}{h_p}(m_j + \pi h_j - m_p)} a_j(\pi) - e^{\frac{il}{h_p}(m_j - \pi h_j - m_p)} a_j(-\pi)\right] + O\left(\frac{1}{l^2}\right).$$

Hence, each of the terms in the two summations in the RHS of (14.3a) (note that G_p also contains a summation) decays exponentially unless $j = p \pm 1$, where there is linear decay. Also the term $j = p$ in G_p oscillates. Clearly, if $a_j(\pm\pi) = 0$, then the decay is quadratic. Similar considerations are valid for the RHS of (14.3b). ∎

Remark 14.1. Taking into consideration that $l \in \mathbb{R}^+$, it follows that the expressions in the LHS of (14.3), taken together, define the Fourier transform of $u_p(s)$.

Remark 14.2. Regarding the choice of k defined by (14.10) we note that $\hat{q}_p(k)$ involves the exponential function $\exp[-ikh_p s]$. In order to define the Fourier transform of $u_p(s)$ we need an oscillatory exponential, and thus we must choose k such that $kh_p \in \mathbb{R}$, i.e.,

$$\arg k = -\arg(h_p) \quad \text{or} \quad \arg k = \pi - \arg(h_p).$$

Among these two permissible choices, it is only the second choice that makes the RHS of (14.3) *bounded* as $l \to \infty$. In this respect we also note that in order to maximize the decay of the RHS of (14.3) as $l \to \infty$, we choose the parametrization defined by (14.8), instead of the parametrization

$$z(s) = z_j + 2sh_j, \quad 0 < s < \pi.$$

14.1.1. The Unknown Values at the Corners

Suppose that the given functions $g_j(s)$ satisfy appropriate compatibility conditions at the corners such that the derivatives of q with respect to z and $\bar{z}$ are continuous at the corners. In this case, these latter values can be determined explicitly, provided that

$$\sin\left(\delta_{j+1} - \delta_j\right) \neq 0, \quad \delta_j = \arg(h_j) - \beta_j, \quad j = 1, \ldots, n. \tag{14.13a}$$

The relevant formulae are

$$u_j(\pi) = \frac{|h_{j+1}| \cos(\delta_{j+1} - \delta_j) g_j(\pi) - |h_j| g_{j+1}(-\pi)}{|h_{j+1}| \sin(\delta_{j+1} - \delta_j)} \tag{14.13b}$$

and

$$u_j(-\pi) = \frac{|h_j| g_{j-1}(\pi) - |h_{j-1}| \cos(\delta_j - \delta_{j-1}) g_j(-\pi)}{|h_{j-1}| \sin(\delta_j - \delta_{j-1})}, \quad j = 1, \ldots, n. \tag{14.13c}$$

Indeed, for the side S_j, (14.7) with

$$dz = h_j ds = |h_j| e^{i \arg(h_j)} ds$$

implies

$$q_z = \frac{e^{-i\delta_j}}{2|h_j|} \left[g_j(s) + i u_j(s) \right], \quad s \in S_j. \tag{14.14}$$

The condition that q_z is continuous at the corner z_j implies that the expression in (14.14) with j replaced by $j - 1$ evaluated at $s = \pi$ (the right end of the side S_{j-1}) equals the expression in (14.14) evaluated at $s = -\pi$ (the left end of the side S_j), i.e.,

$$\frac{e^{-i\delta_{j-1}}}{|h_{j-1}|} \Big[g_{j-1}(\pi) + i u_{j-1}(\pi) \Big] = \frac{e^{-i\delta_j}}{|h_j|} \Big[g_j(-\pi) + i u_j(-\pi) \Big].$$

Similarly, the continuity of $q_{\bar{z}}$ at z_j implies

$$\frac{e^{i\delta_{j-1}}}{|h_{j-1}|}\Big[g_{j-1}(\pi) - iu_{j-1}(\pi)\Big] = \frac{e^{i\delta_j}}{|h_j|}\Big[g_j(-\pi) - iu_j(-\pi)\Big].$$

Solving the above two equations for $u_{j-1}(\pi)$ and $u_j(-\pi)$ and then letting $j \to j+1$ in the expression for $u_{j-1}(\pi)$, we find (14.13b) and (14.13c).

14.1.2. Unknown Functions which Vanish at the Corners

By subtracting the known values at the two corners from the unknown function $u_p(s)$, it follows that it is possible to express the unknown functions in terms of some new unknown functions, denoted by $\check{u}_p(s)$, which *vanish* at the corners:

$$u_p(s) = \check{u}_p(s) + u_{*p}(s) \tag{14.15a}$$

with

$$u_{*p}(s) = \frac{1}{2\pi}\left[(s+\pi)u_p(\pi) - (s-\pi)u_p(-\pi)\right], \quad p = 1, \dots, n. \tag{14.15b}$$

The unknown functions $\check{u}_p(s)$ satisfy equations similar to (14.3) but with G_p, $\tilde{G}_p$ replaced by

$$G_p + U_{*p}, \quad \tilde{G}_p + \tilde{U}_{*p},$$

where the known functions U_{*p} and $\tilde{U}_{*p}$ are defined by

$$U_{*p}(l) = i\sum_{j=1}^{n} E_{jp}(l)\int_{-\pi}^{\pi} e^{il\frac{h_j}{h_p}s} u_{*j}(s)ds$$

and

$$\tilde{U}_{*p}(l) = -i\sum_{j=1}^{n} \bar{E}_{jp}(l)\int_{-\pi}^{\pi} e^{-il\frac{\bar{h}_j}{\bar{h}_p}s} u_{*j}(s)ds.$$

By computing the integrals involving $u_{*j}(s)$ we find that for $p = 1, \dots, n$,

$$U_{*p}(l) = i\sum_{j=1}^{n} E_{jp}(l)\left\{\Big[u_j(\pi) - u_j(-\pi)\Big]\left[\frac{h_p}{ilh_j}\cos\left(\frac{l\pi h_j}{h_p}\right) + \frac{i}{\pi}\frac{h_p^2}{l^2h_j^2}\sin\left(\frac{l\pi h_j}{h_p}\right)\right]\right.$$
$$\left. + \frac{h_p}{lh_j}\Big[u_j(\pi) + u_j(-\pi)\Big]\sin\left(\frac{l\pi h_j}{h_p}\right)\right\} \tag{14.16a}$$

and

$$\tilde{U}_{*p}(l) = -i\sum_{j=1}^{n} \bar{E}_{jp}(l)\left\{\Big[u_j(\pi) - u_j(-\pi)\Big]\left[\frac{\bar{h}_p}{-il\bar{h}_j}\cos\left(\frac{l\pi\bar{h}_j}{\bar{h}_p}\right) - \frac{i}{\pi}\frac{\bar{h}_p^2}{l^2\bar{h}_j^2}\sin\left(\frac{l\pi\bar{h}_j}{\bar{h}_p}\right)\right]\right.$$
$$\left. + \frac{\bar{h}_p}{l\bar{h}_j}\Big[u_j(\pi) + u_j(-\pi)\Big]\sin\left(\frac{l\pi\bar{h}_j}{\bar{h}_p}\right)\right\}, \quad l \in \mathbb{R}^+. \tag{14.16b}$$

Remark 14.3. The function $\check{u}_p(s)$, which is defined for $-\pi < s < \pi$, vanishes at the endpoints. Thus, a convenient representation for such a function is an expansion in terms of the modified sine-Fourier series,

$$\check{u}_p(s) = \sum_{m=1}^{\infty} \left[s_m^p \sin(ms) + c_m^p \cos\left(m - \frac{1}{2}\right) s \right], \tag{14.17a}$$

where the constants s_m^p and c_m^p are defined by

$$s_m^p = \frac{1}{\pi} \int_{-\pi}^{\pi} \check{u}_p(s) \sin(ms) ds, \quad m = 1, 2, \ldots, \tag{14.17b}$$

and

$$c_m^p = \frac{1}{\pi} \int_{-\pi}^{\pi} \check{u}_p(s) \cos\left(m - \frac{1}{2}\right) s ds, \quad m = 1, 2, \ldots, \quad p = 1, \ldots, n. \tag{14.17c}$$

The advantage of the above expansion is that s_m^p and c_m^p are of order $1/m^2$ as $m \to \infty$, provided that $\check{u}_p(s)$ has sufficient smoothness. The representation (14.17a) can be obtained by starting with the usual sine-Fourier series in the interval $(0, \pi)$ and then using a change of variables to map this interval to $(-\pi, \pi)$. The analogue of the representation (14.17a) corresponding to the cosine-Fourier series was introduced in [94].

Proposition 14.2. Let q satisfy the boundary value problem specified in Proposition 14.1. Assume that the values of the unknown functions $\{u_j\}_1^n$ at the corners $\{z_j\}_1^n$ are given by (14.13). Express $\{u_j\}_1^n$ in terms of the unknown functions $\{\check{u}_j\}_1^n$ by (14.15), and approximate the latter functions by $\check{u}_j^N(s)$, where

$$\check{u}_p^N(s) = \sum_{m=1}^{N} \left[s_m^p \sin(ms) + c_m^p \cos\left(m - \frac{1}{2}\right) s \right], \quad p = 1, \ldots, n. \tag{14.18}$$

Then the constants s_m^p and c_m^p, $m = 1, \ldots, N$, $p = 1, \ldots, n$, satisfy the following $2Nn$ algebraic equations:

$$\begin{aligned} 2\pi s_m^p = i \sum_{\substack{j=1 \\ j \neq p}}^{n} \Bigg\{ & \sum_{n'=1}^{N} s_{n'}^j \left[E_{jp}(m) S_{jp}^{n'}(m) - \bar{E}_{jp}(m) \bar{S}_{jp}^{n'}(m) \right] \\ & + \sum_{n'=1}^{N} c_{n'}^j \left[E_{jp}(m) C_{jp}^{n'}(m) - \bar{E}_{jp}(m) \bar{C}_{jp}(m) \right] \Bigg\} \\ & + G_p(m) + \tilde{G}_p(m) + U_{*p}(m) + \tilde{U}_{*p}(m) \end{aligned} \tag{14.19a}$$

and

$$\begin{aligned} 2\pi c_m^p = - \sum_{\substack{j=1 \\ j \neq p}}^{n} \Bigg\{ & \sum_{n'=1}^{N} s_{n'}^j \left[E_{jp}\left(m - \frac{1}{2}\right) S_{jp}^{n'}\left(m - \frac{1}{2}\right) + \bar{E}_{jp}\left(m - \frac{1}{2}\right) \bar{S}_{jp}^{n'}\left(m - \frac{1}{2}\right) \right] \\ & + \sum_{n'=1}^{N} c_{n'}^j \left[E_{jp}\left(m - \frac{1}{2}\right) C_{jp}^{n'}\left(m - \frac{1}{2}\right) + \bar{E}_{jp}\left(m - \frac{1}{2}\right) C_{jp}^{n'}\left(m - \frac{1}{2}\right) \right] \Bigg\} \end{aligned}$$

$$+ i\left[G_p\left(m-\frac{1}{2}\right) - \tilde{G}_p\left(m-\frac{1}{2}\right) + U_{*p}\left(m-\frac{1}{2}\right) - \tilde{U}_{*p}\left(m-\frac{1}{2}\right)\right], \tag{14.19b}$$

where the known functions G_p, $\tilde{G}_p$, U_{*p}, $\tilde{U}_{*p}$ are defined by (14.4), (14.16), and

$$S^n_{jp}(m) = \frac{2in(-1)^{n-1}\sin\left(\frac{m\pi h_j}{h_p}\right)}{n^2 - \frac{m^2h_j^2}{h_p^2}}, \quad C^n_{jp}(m) = \frac{2\left(n-\frac{1}{2}\right)(-1)^{n-1}\cos\left(\frac{m\pi h_j}{h_p}\right)}{\left(n-\frac{1}{2}\right)^2 - \frac{m^2h_j^2}{h_p^2}},$$

$$j = 1,\ldots,n, \quad p = 1,\ldots,n, \quad m = 1,\ldots,N. \tag{14.20}$$

Proof. Equation (14.18) implies

$$s^p_m = \frac{1}{\pi}\int_{-\pi}^{\pi} \breve{u}^N_p(s)\sin(ms)ds. \tag{14.21}$$

Recall that the functions $\breve{u}_p$ satisfy equations similar to (14.3) but with G_p and $\tilde{G}_p$ replaced by $G_p + U_{*p}$ and $\tilde{G}_p + \tilde{U}_{*p}$. Replacing the function $\breve{u}_p$ with $\breve{u}^N_p$ defined in (14.18) in the equations satisfied by $\breve{u}_p$, subtracting these equations, and evaluating the resulting equation at $l = m$, we find (14.19a). In this respect we note that the LHS of the resulting equation immediately yields s^p_m, whereas for the evaluation of the RHS of the resulting equations we use the expression

$$\begin{aligned}&\int_{-\pi}^{\pi} e^{im\frac{h_j}{h_p}s} u^N_j(s)ds \\ &\quad = \sum_{n=1}^{N}\left\{s^j_n\int_{-\pi}^{\pi} e^{im\frac{h_j}{h_p}}\sin(ns)ds + c^j_n\int_{-\pi}^{\pi} e^{im\frac{h_j}{h_p}}\cos\left(n-\frac{1}{2}\right)sds\right\}\end{aligned} \tag{14.22}$$

and then we evaluate the above integrals explicitly.

Proceeding as earlier, where we now add the equations satisfied by $\{\breve{u}^N_j\}^n_1$, and then evaluating the resulting equation at $l = m - \frac{1}{2}$, we find (14.19b). ∎

Remark 14.4. The LHSs of equations (14.3) involve integrals of $u_p(s)$ with respect to the exponential functions $\exp[\pm ils]$. This suggests that $u_p(s)$ should be represented in terms of a Fourier-type expansion. Indeed, if we ignore for a moment the first terms in the RHSs of equations (14.3), then by expanding $u_p(s)$ in terms of a Fourier-type expansion, such as (14.17a), the relevant Fourier coefficients can be immediately obtained in terms of G_p and $\tilde{G}_p$. Thus, the choice of the representation (14.18) is consistent with the fact that $\breve{u}_p(s)$ vanishes at the corners, and the evaluation of the global relations at $l = m$ and $l = m - \frac{1}{2}$ is consistent with the orthogonality conditions associated with this expansion (see (14.17b) and (14.17c)). In order to improve convergence, one could use an expansion in terms of Chebyshev polynomials, but since the associated orthogonality conditions do *not* involve $\exp[\pm ils]$, the diagonal blocks of the associated linear system are *not* the identity matrix but are full matrices.

14.2 The Modified Helmholtz Equation

For simplicity we consider the Dirichlet problem; other boundary value problems can be treated in a similar manner.

Proposition 14.3. Let the complex-valued function $q(z, \bar{z})$ satisfy the modified Helmholtz equation (50) in the interior of the convex polygon described in Proposition 14.1 and let q satisfy Dirichlet boundary conditions on each side, namely

$$q_j(s) = d_j(s), \quad j = 1, \ldots, n, \tag{14.23}$$

where s parametrizes the side S_j, q_j denotes q on this side, and the given complex-valued functions $\{d_j\}_1^n$ have sufficient smoothness. Let $u_j(s)$ denote the Neumann boundary value on the side S_j. The n unknown complex-valued functions $\{u_j\}_1^n$ satisfy the following $2n$ equations for all $l \in \mathbb{R}^+$ and $p = 1, \ldots, n$:

$$\int_{-\pi}^{\pi} e^{ils} u_p(s)ds = -\sum_{\substack{j=1 \\ j\neq p}}^{n} E_{jp}(k_p(l)) \int_{-\pi}^{\pi} e_j(k_p(l), s)u_j(s)ds - G_p(l) \tag{14.24a}$$

and

$$\int_{-\pi}^{\pi} e^{-ils} u_p(s)ds = -\sum_{\substack{j=1 \\ j\neq p}}^{n} E_{jp}(\bar{k}_p(l)) \int_{-\pi}^{\pi} \overline{e_j(\bar{k}_p(l), s)}u_j(s)ds - \tilde{G}_p(l), \tag{14.24b}$$

where the exponential functions $E_{jp}(k)$ and $e_j(k, s)$ are defined by

$$E_{jp}(k) = e^{-i\beta(m_j - m_p)k + \frac{i\beta}{k}(\bar{m}_j - \bar{m}_p)}, \quad e_j(k, s) = e^{-i\beta\left(kh_j - \frac{\bar{h}_j}{k}\right)s},$$
$$j = 1, \ldots, n \quad p = 1, \ldots, n, \quad k \in \mathbb{C}, \quad -\pi < s < \pi, \tag{14.25}$$

the function $k_p(l)$ is defined by

$$k_p(l) = -\frac{l + \sqrt{l^2 + 4\beta^2|h_p|^2}}{2\beta h_p}, \quad p = 1, \ldots, n, \quad l \in \mathbb{R}^+, \tag{14.26}$$

and the known functions $G_p(l)$ and $\tilde{G}_p(l)$ are given by

$$G_p(l) = \sum_{j=1}^{n} E_{jp}(k_p(l))P_j(k_p(l)) \int_{-\pi}^{\pi} e_j(k_p(l), s)g_j(s)ds, \tag{14.27a}$$

$$\tilde{G}_p(l) = \sum_{j=1}^{n} \overline{E_{jp}(\bar{k}_p(l))}\; \overline{P_j(\bar{k}_p(l))} \int_{-\pi}^{\pi} \overline{e_j(\bar{k}_p(l), s)}g_j(s)ds, \quad p = 1, \ldots, n, \quad l \in \mathbb{R}^+, \tag{14.27b}$$

with

$$P_j(k) = \beta\left(\frac{\bar{h}_j}{k} + kh_j\right), \quad j = 1, \ldots, n, \quad k \in \mathbb{C}. \tag{14.27c}$$

Each of the terms appearing in the summations on the RHSs of equations (14.24) decays exponentially as $l \to \infty$, except for the terms with $j = p$ in G_p and $\tilde{G}_p$ which oscillate, as well as the terms with $j = p \pm 1$ which decay linearly.

Proof. Using the parametrization defined by (14.8) in the definition of $\hat{q}_j(k)$ (see (51d)), we find

$$\hat{q}_j(k) = ie^{-i\beta\left(m_j k - \frac{\bar{m}_j}{k}\right)} \int_{-\pi}^{\pi} e_j(k,s)\left[u_j(s) + P_j(k)g_j(s)\right] ds.$$

Substituting this expression in the first of the global relations (52) of the introduction and then evaluating the resulting equation at $k = k_p(l)$ we find (14.24a). Recalling that the second of the global relations in (52) can be obtained from the first global relation by taking the Schwarz conjugate of all terms except of q, (14.24a) immediately implies (14.24b).

The reasons for choosing k_p to satisfy (14.26) are similar to those discussed in section 14.1. In particular, in order to obtain the Fourier transform of $u_p(s)$ we choose

$$-\beta\left(kh_p - \frac{\bar{h}_p}{k}\right) = l, \quad l \in \mathbb{R}^+.$$

This yields two possible choices for k_p, namely $k_p^{\pm}$, where

$$k_p^{\pm} = \frac{-l \pm \sqrt{l^2 + 4\beta^2|h_p|^2}}{2h_p\beta}$$

and we choose the negative root so that $\arg k_p = \pi - \arg h_j$ (recall that the rays l_p in the integral representation of q are *identical* to those appearing in the integral representation of the Laplace equation).

Also,

$$E_{jp}(k_p(l))e_j(k_p(l),s) = e^{i\beta L_p\left(\frac{m_j - m_p + h_j s}{h_p}\right)} e^{-\frac{i\beta|h_p|}{L_p}\left(\frac{\bar{m}_j - \bar{m}_p + \bar{h}_j s}{\bar{h}_p}\right)}, \qquad (14.28)$$

where the positive constant L_p is defined by

$$L_p = \frac{l + \sqrt{l^2 + 4\beta^2|h_p|^2}}{2\beta}. \qquad (14.29)$$

Using the estimate given by (14.11) and noting that the terms appearing in the RHS of (14.28) can be treated in a similar way as the exponential term appearing in (14.12), we find that the RHSs of equations (14.24) as $l \to \infty$ have a similar behavior as the RHSs of equations (14.3), where now in the case of $j = p \pm 1$, the relevant decay is of order $O(1/L_p)$ which equals $O(1/l)$.

14.2.1. The Unknown Values at the Corners

Suppose that the given functions $d_j(s)$ satisfy appropriate compatibility conditions so that the derivatives of q with respect to z and $\bar{z}$ are continuous. Then proceeding as with the Laplace equation we find

$$u_j(\pi) = \frac{|h_{j+1}|\cos(\alpha_{j+1} - \alpha_j)\frac{d}{ds}d_j(\pi) - |h_j|\frac{d}{ds}d_{j+1}(-\pi)}{|h_{j+1}|\sin(\alpha_{j+1} - \alpha_j)}, \quad \alpha_j = \arg h_j, \qquad (14.30a)$$

$$u_j(-\pi) = \frac{|h_j|\frac{d}{ds}d_{j-1}(\pi) - h_{j-1}\cos(\alpha_j - \alpha_{j-1})\frac{d}{ds}d_j(-\pi)}{|h_{j-1}|\sin(\alpha_j - \alpha_{j-1})}, \quad j = 1, \ldots, n. \quad (14.30b)$$

Convexity implies

$$\alpha_j < \alpha_{j+1} < \alpha_j + \pi, \quad j = 1, \ldots, n,$$

and thus

$$\sin(\alpha_{j+1} - \alpha_j) \neq 0.$$

14.2.2. Unknown Functions which Vanish at the Corners

Proceeding as with Laplace's equation we define the function $\check{u}_p(s)$ by (14.15a), and then $U_{*p}(l)$ and $\tilde{U}_{*p}(l)$ are given by

$$U_{*p}(l) = \sum_{j=1}^{n} E_{jp}(k_p(l)) \int_{-\pi}^{\pi} e_j(k_p(l), s) u_{*j}(s) ds$$

and

$$U_{*p}(l) = \sum_{j=1}^{n} \overline{E_{jp}(\bar{k}_p(l))} \int_{-\pi}^{\pi} \overline{e_j(\bar{k}_p(l), s)} u_{*p}(s) ds.$$

Hence, instead of the function $\exp[ilh_j s/h_p]$ of section 14.1, we now have the function

$$e_j(k_p(l), s) = e^{i\beta\left(\frac{L_p h_j}{h_p} - \frac{\bar{h}_j h_p}{L_p}\right)s},$$

where L_p is defined by (14.29). Hence, the formulae for U_{*p} can be obtained from the formulae (14.16) by replacing $E_{jp}(l)$ with $E_{jp}(k(l))$ as well as replacing lh_j/h_p with

$$\beta\left(\frac{L_p h_j}{h_p} - \frac{\bar{h}_j h_p}{L_p}\right).$$

Hence, $U_{*p}(l)$ and $\tilde{U}_{*p}(l)$ are defined by the following equations for $p = 1, \ldots, n, l \in \mathbb{R}^+$:

$$\begin{aligned} U_{*p}(l) = i \sum_{j=1}^{n} E_{jp}(k_p(l)) \Bigg\{ \Big[u_j(\pi) - u_j(-\pi)\Big]\Big[-\frac{i}{H_{jp}(l)} \cos(\pi H_{jp}(l)) \\ + \frac{i}{\pi (H_{jp}(l))^2} \sin(\pi H_{jp}(l))\Big] + \frac{1}{H_{jp}(l)} \Big[u_j(\pi) + u_j(-\pi)\Big] \sin\left(\pi H_{jp}(l)\right) \Bigg\} \end{aligned} \quad (14.31a)$$

and

$$\begin{aligned} \tilde{U}_{*p}(l) = -i \sum_{j=1}^{n} \overline{E_{jp}(\bar{k}_p(l))} \Bigg\{ \Big[u_j(\pi) - u_j(-\pi)\Big]\Big[\frac{i}{\bar{H}_{jp}(l)} \cos(\pi \bar{H}_{jp}(l)) \\ - \frac{i}{\pi (\bar{H}_{jp}(l))^2} \sin(\pi \bar{H}_{jp}(l))\Big] + \frac{1}{\bar{H}_{jp}(l)} \Big[u_j(\pi) + u_j(-\pi)\Big] \sin\left(\pi \bar{H}_{jp}(l)\right) \Bigg\}, \end{aligned} \quad (14.31b)$$

with

$$H_{jp}(l) = \frac{1}{2}\frac{h_j}{h_p}\left(l + \sqrt{l^2 + 4\beta^2 |h_p|^2}\right) - \frac{2\beta^2 \bar{h}_j h_p}{l + \sqrt{l^2 + 4\beta^2 |h_p|^2}}, \quad j = 1, \ldots, n. \quad (14.31c)$$

Proposition 14.4. Let q satisfy the Dirichlet boundary value problem specified in Proposition 14.3. Assume that the unknown Neumann boundary values at the corner $\{z_j\}_1^n$ are given by (14.30). Express the unknown Neumann boundary values $\{u_j\}_1^n$ in terms of the unknown functions $\{\breve{u}_j\}_1^n$ by (14.15) and approximate the latter functions by the functions $\{\breve{u}_j^N\}_1^n$ defined in (14.18). Then the constants s_m^p and c_m^p, $m = 1, \dots, N$, $p = 1, \dots, n$, satisfy the following $2Nn$ algebraic equations:

$$2\pi s_m^p = i \sum_{\substack{j=1 \\ j\neq p}}^{n} \left\{ \sum_{n'=1}^{N} s_{n'}^j \left[E_{jp}(k_p(m)) S_{jp}^{n'}(m) - \overline{E_{jp}(\bar{k}_p(m))} \bar{S}_{jp}^{n'}(m) \right] \right.$$
$$\left. + \sum_{n'=1}^{N} c_{n'}^j \left[E_{jp}(k_p(m)) C_{jp}^{n'}(m) - \overline{E_{jp}(\bar{k}_p(m)} \bar{C}_{jp}^{n'}(m) \right] \right\}$$
$$+ iG_p(m) - i\tilde{G}_p(m) + iU_{*p}(m) - i\tilde{U}_{*p}(m) \tag{14.32a}$$

and

$$2\pi c_m^p = - \sum_{\substack{j=1 \\ j\neq p}}^{n} \left\{ \sum_{n'=1}^{N} s_{n'}^j \left[E_{jp}\left(k_p\left(m-\frac{1}{2}\right)\right) S_{jp}^{n'}\left(m-\frac{1}{2}\right) \right. \right.$$
$$\left. + \overline{E_{jp}\left(\bar{k}_p\left(m-\frac{1}{2}\right)\right)} \bar{S}_{jp}^{n'}\left(m-\frac{1}{2}\right) \right]$$
$$+ \sum_{n'=1}^{N} c_{n'}^j \left[E_{jp}\left(k_p\left(m-\frac{1}{2}\right)\right) C_{jp}^{n'}\left(m-\frac{1}{2}\right) \right. \tag{14.32b}$$
$$\left. \left. + \overline{E_{jp}\left(\bar{k}_p\left(m-\frac{1}{2}\right)\right)} \bar{C}_{jp}^{n'}\left(m-\frac{1}{2}\right) \right] \right\}$$
$$- G_p(m) - \tilde{G}_p(m) - U_{*p}(m) - \tilde{U}_{*p}(m),$$

where the known functions G_p, $\tilde{G}_p$, U_{*p}, $\tilde{U}_{*p}$ are defined by (14.27), (14.31), and

$$S_{jp}^n(m) = \frac{2in(-1)^{n-1}\sin\left(H_{jp}(m)\right)}{n^2 - \left(H_{jp}(m)\right)^2}, \quad C_{jp}^n(m) = \frac{2\left(n-\frac{1}{2}\right)(-1)^{n-1}\cos\left(H_{jp}(m)\right)}{\left(n-\frac{1}{2}\right)^2 - \left(H_{jp}(m)\right)^2},$$

with $H_{jp}(m)$ defined by (14.31c).

Proof. The proof is similar to that of Proposition 14.2. ∎

14.3 Further Developments and Numerical Computations

The main ideas of the technique presented in section 14.1 were introduced in [92]. However, although the values of k were correctly chosen to be those in (14.10), the global relations were evaluated at $l = m$ instead of $l = m$, and/or $l = m - \frac{1}{2}$. As a result, the relevant linear system possesses a large condition number and numerical computations performed in [92] suggest linear convergence. In [39], the choice of $l = m$ and/or $l = m - \frac{1}{2}$ led to a linear system with a small condition number. Also the numerical computations in [39] suggest quadratic convergence for the modified sine-Fourier series (see (14.18)) as well as exponential convergence for the Chebyshev basis. The extension of the technique to modified Helmholtz and Helmholtz equations is presented in [40]. Regarding the latter equation we note that now k must be chosen to lie on parts of certain rays as well as on parts of certain circular arcs. This is consistent with the fact that the contour in the complex k-plane associated with the solution of the Helmholtz equation involves rays and circular arcs; see [22]. Details of the application of the collocation method to *regular* polygons can be found in [41].

Remark 14.5. The following result has been derived by E.A. Spence [24] using the techniques of [96]. Suppose that $u(s) \in C^3(-\pi, \pi)$ and that $u(\pi) = u(-\pi) = 0$. Define s_m and c_m by (14.17b) and (14.17c), where $\check{u}_p(s)$ is replaced by $u(s)$. Define the function $u^N(s)$ by the RHS of (14.18), where s_m^p and c_m^p are replaced by s_m and c_m. Then

$$\|u(s) - u^N(s)\|_\infty = O\left(\frac{1}{N^2}\right).$$

Combining this result with a theorem of [97] on infinite matrices, one should be able to prove the quadratic convergence of the approximations of Propositions 14.2 and 14.4.

Remark 14.6. It is important to emphasize that the global relation is valid for *all* complex k. The restriction of k on the rays (14.10) and (14.26) was imposed in order to express the unknown function in terms of Fourier integrals. It appears that an alternative effective approach is to obtain an *overdetermined* system of unknowns by choosing for k a sufficiently large set of *arbitrary* finite complex values [42].

In what follows we present some numerical results from [40].

14.3.1. Numerical Examples

In order to illustrate the numerical implementation of the new collacation method to the Laplace and modified Helmholtz equations, we will consider a variety of regular and irregular polygons.

We will study the Laplace equation with the exact solution

$$q(z, \bar{z}) = e^{3z} + 2e^{3\bar{z}}, \tag{14.33}$$

and the modified Helmholtz equation with $\beta = 10$ and the exact solution

$$q(z, \bar{z}) = e^{11z + \frac{100}{11}\bar{z}}. \tag{14.34}$$

Analytic expressions for the known boundary functions $\{g_j(s)\}_{j=1}^n$, $\{d_j(s)\}_{j=1}^n$ and the unknown boundary data $\{u_j(s)\}_{j=1}^n$ can be easily computed from (14.33) and (14.34).

To demonstrate the performance of the method, we use the discrete maximum relative error

$$E_\infty := \frac{||u - u^N||_\infty}{||u_n||_\infty}, \tag{14.35}$$

Table 14.1. *Vertices of irregular polygons prior to rotation.*

Triangle	$\left(-4,-\frac{6}{5}\right)$, $\left(-1,-\frac{2\sqrt{24}}{25}\right)$, $\left(3,\frac{8}{5}\right)$
Square	$\left(1,\frac{2\sqrt{24}}{25}\right)$, $\left(-4,-\frac{6}{5}\right)$, $\left(4,-\frac{6}{5}\right)$, $\left(4,\frac{6}{5}\right)$
Pentagon	$(0,2)$, $(-5,0)$, $\left(-2,-\frac{2\sqrt{21}}{25}\right)$, $\left(4,-\frac{6}{5}\right)$, $\left(3,\frac{8}{5}\right)$
Hexagon	$\left(1,\frac{2\sqrt{24}}{25}\right)$, $\left(-\frac{9}{2},\frac{2\sqrt{19}}{10}\right)$, $\left(-4,-\frac{6}{5}\right)$, $\left(-1,-\frac{2\sqrt{24}}{25}\right)$, $\left(2,-\frac{2\sqrt{21}}{25}\right)$, $\left(\frac{9}{2},\frac{2\sqrt{19}}{10}\right)$
Octagon	$\left(1,\frac{2\sqrt{24}}{25}\right)$, $\left(-2,\frac{2\sqrt{21}}{25}\right)$, $\left(-3,\frac{8}{5}\right)$, $(-5,0)$, $\left(-4,-\frac{6}{5}\right)$, $\left(-1,-\frac{2\sqrt{24}}{25}\right)$, $\left(2,-\frac{2\sqrt{21}}{25}\right)$, $\left(3,\frac{8}{5}\right)$

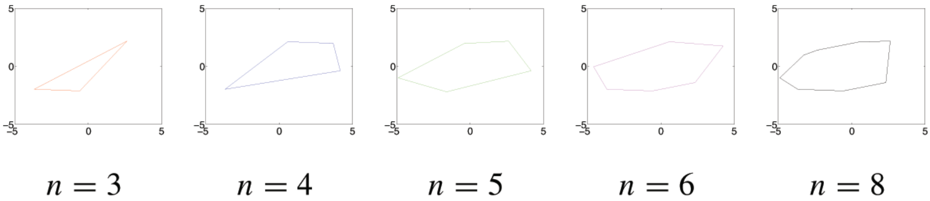


Figure 14.4. *Irregular polygons.*

where

$$u_j^N(s) = \check{u}_j^N(s) + u_{*j}(s), \quad -\pi \le s \le \pi,\ 1 \le j \le n, \tag{14.36}$$

$$||u||_\infty = \max_{1\le j\le n}\left\{\max_{s\in S}|u_n^{(j)}(s)|\right\}. \tag{14.37}$$

We consider 10001 evenly spaced points s_i,

$$S = \{s_i\}_{i=1}^{10001} \subset [-\pi, \pi], \tag{14.38}$$

with the points s_i, $-\pi = s_1 < s_2 < \cdots < s_{10000} < s_{10001} = \pi$, given by

$$s_i = \pi\left[-1 + \frac{2(i-1)}{10000}\right], \quad 1 \le i \le 10001. \tag{14.39}$$

We consider regular polygons with $n = 3,\ 4,\ 5,\ 6,\ 8$ sides, whose vertices lie on the circle centered at the origin with radius $\sqrt{2}$ in the complex plane (with a vertex on the positive real axis). These polygons are then rotated through an angle of $-\frac{1}{5}$ about the origin to avoid nongeneric results due to alignment with the coordinate axes. Thus, we consider polygons with the vertices

$$z(j) = \sqrt{2}e^{i\left[2(j-1)\frac{\pi}{n}-\frac{1}{5}\right]}, \quad 1 \le j \le n. \tag{14.40}$$

We also consider irregular polygons with $n = 3,\ 4,\ 5,\ 6,\ 8$ sides, whose vertices lie on the ellipse $\left(\frac{x}{5}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$ in the complex plane rotated through an angle of $\frac{1}{5}$ about the origin. The x and y coordinates of the vertices of the polygons before rotation are given (in an anticlockwise direction) in Table 14.1 and the polygons are shown in Figure 14.4.

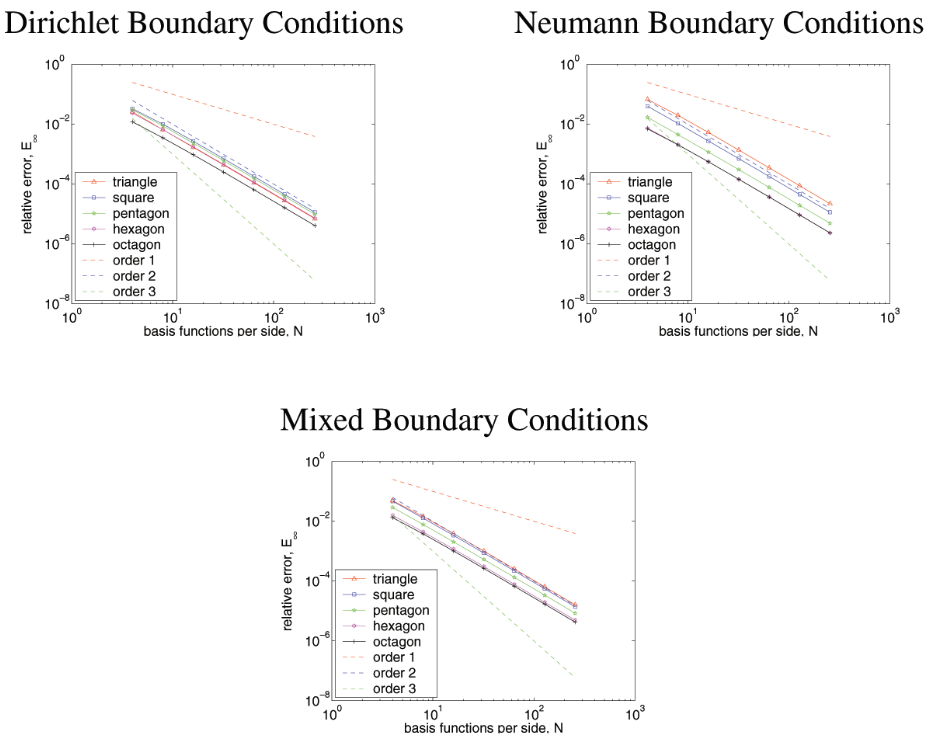


Figure 14.5. *E_∞ as a function of N for the Laplace equation in regular polygons.*

Figures 14.5 and 14.6 refer to the Laplace equation with Dirichlet ($\beta_j = 0$), Neumann ($\beta_j = \frac{\pi}{2}$), and mixed ($\beta_j = \frac{\pi}{3}$) boundary conditions in the regular and irregular polygons, respectively. The red, blue, and green dotted lines are the lines $\frac{1}{N}$, $\frac{1}{N^2}$, and $\frac{1}{N^3}$, indicating the slopes for first, second, and third order convergence, respectively. The error lines are all asymptotically parallel to the $\frac{1}{N^2}$ line, indicating a quadratically convergent method with respect to the discrete maximum relative error. To highlight this, the order of convergence (O.o.C.) has been estimated for the triangles in Tables 14.2 and 14.3.

It can be seen from Figure 14.7 that the condition numbers of the associated matrices are small and grow only very slowly with N.

In complete analogy, Figures 14.8 and 14.9 and Table 14.4 refer to the modified Helmholtz equation.

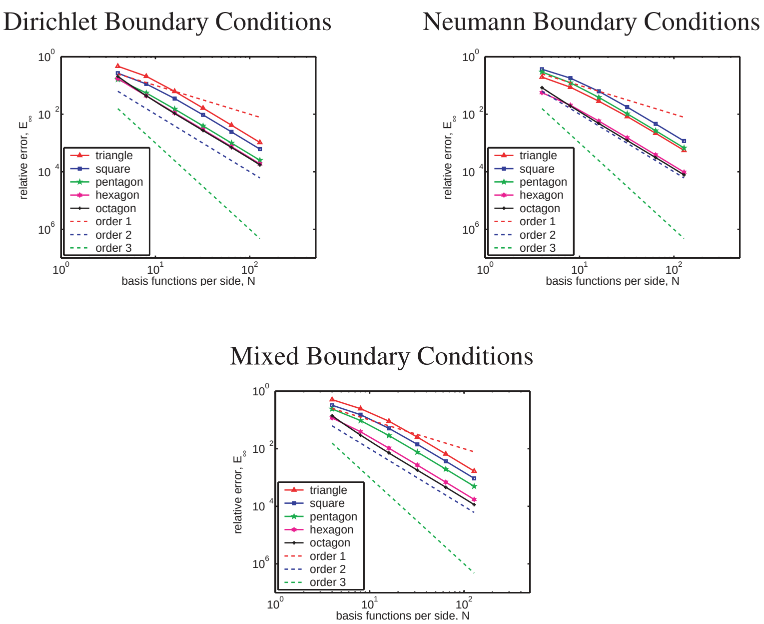


Figure 14.6. *E_∞ as a function of N for the Laplace equation in irregular polygons.*

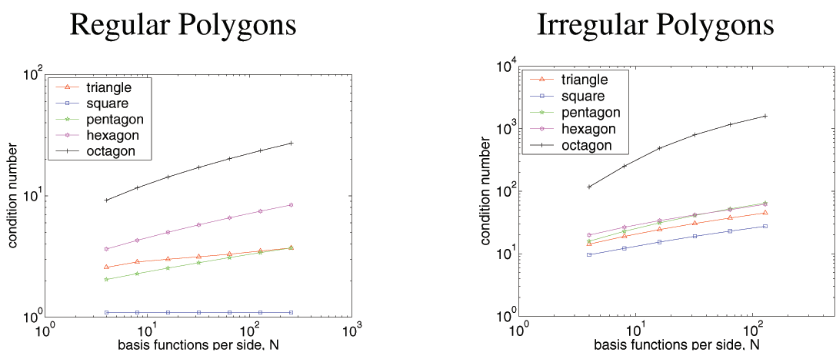


Figure 14.7. *The condition number of the coefficient matrix as a function of N for the Laplace equation.*

Table 14.2. *Equilateral triangle, Laplace equation.*

	Dirichlet BCs		Neumann BCs		Mixed BCs	
N	E_∞	O.o.C.	E_∞	O.o.C.	E_∞	O.o.C.
4	2.6019e-02	—	6.7659e-02	—	4.8884e-02	—
8	6.5908e-03	1.9811	1.9896e-02	1.7658	1.4438e-02	1.7595
16	1.7026e-03	1.9527	5.3327e-03	1.8995	3.8872e-03	1.8931
32	4.3592e-04	1.9656	1.3777e-03	1.9526	1.0057e-03	1.9505
64	1.1048e-04	1.9802	3.4996e-04	1.9770	2.5557e-04	1.9764
128	2.7823e-05	1.9895	8.8177e-05	1.9887	6.4403e-05	1.9885
256	6.9784e-06	1.9953	2.2119e-05	1.9951	1.6156e-05	1.9950

Table 14.3. *Irregular triangle, Laplace equation.*

	Dirichlet BCs		Neumann BCs		Mixed BCs	
N	E_∞	O.o.C.	E_∞	O.o.C.	E_∞	O.o.C.
4	4.6405e-01	—	1.9495e-01	—	5.0830e-01	—
8	2.0730e-01	1.1626	8.6601e-02	1.1706	2.4640e-01	1.0447
16	6.2140e-02	1.7381	2.8274e-02	1.6149	9.0366e-02	1.4471
32	1.6293e-02	1.9313	8.2849e-03	1.7709	2.5391e-02	1.8315
64	4.1470e-03	1.9741	2.1762e-03	1.9287	6.5868e-03	1.9467
128	1.0453e-03	1.9881	5.5307e-04	1.9763	1.6686e-03	1.9809
256	2.6216e-04	1.9954	1.3896e-04	1.9928	4.1881e-04	1.9943

Table 14.4. *Modified Helmholtz equation.*

Equilateral triangle

N	E_∞	O.o.C.
4	4.3435e-01	—
8	2.4280e-01	0.8391
16	9.6685e-02	1.3284
32	2.9334e-02	1.7207
64	7.8274e-03	1.9060
128	1.9983e-03	1.9697
256	5.0311e-04	1.9898

Irregular triangle

N	E_∞	O.o.C.
4	7.2627e-01	—
8	5.9449e-01	0.2889
16	4.1420e-01	0.5213
32	2.2008e-01	0.9123
64	8.3167e-02	1.4039
128	2.4321e-02	1.7738
256	6.3678e-03	1.9334
512	1.6117e-03	1.9822

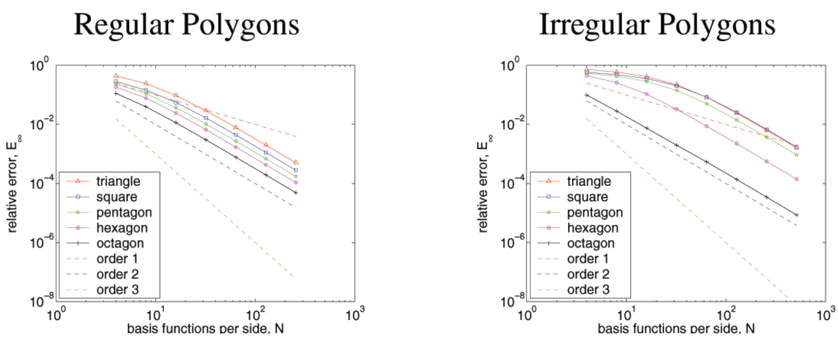


Figure 14.8. E_∞ *as a function of N for the modified Helmholtz equation.*

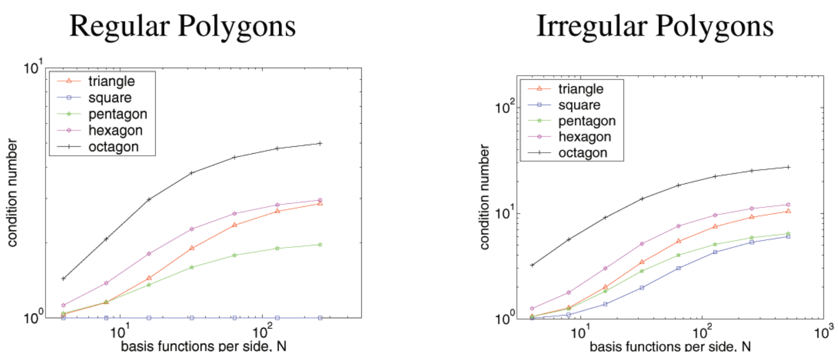


Figure 14.9. *The condition number of the coefficient matrix as a function of N for the modified Helmholtz equation.*

Part V

Integrable Nonlinear PDEs

In this part we present a method which yields an effective nonlinearization of some of the results presented in Chapters 1, 9, and 10. We begin by showing that there exists an algorithmic approach which, starting from a dispersive linear evolution PDE, yields an integrable nonlinear PDE; this approach, which was summarized in section I.5.1 of the introduction, is discussed in Chapter 15. We then concentrate on the analysis of certain integrable nonlinear PDEs. In Chapter 16 we derive integral representations for the following integrable nonlinear PDEs formulated on the half-line: The nonlinear Schrödinger (NLS), the Korteweg–de Vries (KdV), the modified KdV, and the sine-Gordon (sG) equations. The relevant formulae provide nonlinear versions of (1.16) (for the defocusing NLS see equation (92)). These formulae differ from the RHS of (1.16) in two important ways: (a) Instead of involving the transforms of the initial condition and of the boundary values (which are denoted by $\hat{q}_0(k)$ and $\tilde{g}(k)$, respectively, in (1.16)), they involve certain functions denoted by $\{a(k), b(k), A(k), B(k)\}$. These functions, although uniquely defined in terms of the initial condition and of the boundary values, cannot be written explicitly (for the defocusing NLS see (86)–(91)). (b) They involve the entries of the matrix-valued function $M(x, t, k)$ which is the solution of a matrix Riemann–Hilbert (RH) problem uniquely defined in terms of $\{a(k), b(k), A(k), B(k)\}$. In the linear limit (when q is small), the functions $a(k)$ and $A(k)$ approach 1, whereas the functions $b(k)$ and $B(k)$ approach $\hat{q}_0(k)$ and $\tilde{g}(k)$; furthermore, $M(x, t, k)$ approaches diag(1, 1), and hence the solution of an integrable nonlinear PDE reduces to the solution of the corresponding linear PDE given by (1.16). The functions $A(k)$ and $B(k)$ appearing in the above integrable representations are defined in terms of *all* boundary values. Thus, in order to solve a concrete initial-boundary value problem it is first necessary to determine the unknown boundary values in terms of the given initial and boundary conditions. The solution of this problem, i.e., the characterization of the generalized Dirichlet to Neumann map for integrable nonlinear PDEs, is discussed in Chapter 18 (see also the discussion in section I.5.3 of the introduction). There exist certain simple initial-boundary value problems which are called *linearizable*, for which it is possible to solve the global relation *directly* for $A(k)$ and $B(k)$ (without the need to determine the unknown boundary values). The solution of such problems is discussed in Chapter 17. The long-time asymptotics of the NLS is discussed in Chapter 19. In addition, in Chapter 19, an interesting asymptotic limit of a system of three nonlinear PDEs describing the transient stimulated Raman scattering is also discussed. These three nonlinear PDEs belong to a class of integrable PDEs which are distinguished by the fact that *all* boundary values appearing in the representation of the solution are prescribed as boundary conditions. Hence, for this class of PDEs it is possible to solve initial-boundary value problems *without* the need to analyze the associated global relation.

Chapter 15

From Linear to Integrable Nonlinear PDEs

Let $q(x, t)$ satisfy the linear dispersive PDE

$$q_t + iw(-i\partial_x)q = 0, \tag{15.1a}$$

where

$$w(k) \ \text{ is a real polynomial of degree } \ n. \tag{15.1b}$$

It was indicated in section I.5.1 of the introduction that starting with (15.1a) it is possible to construct algorithmically an integrable nonlinear PDE. The four algorithmic steps employed in this construction which are refereed to as (1)–(4) in Diagram 15.1 will be discussed in sections 15.1–15.4.

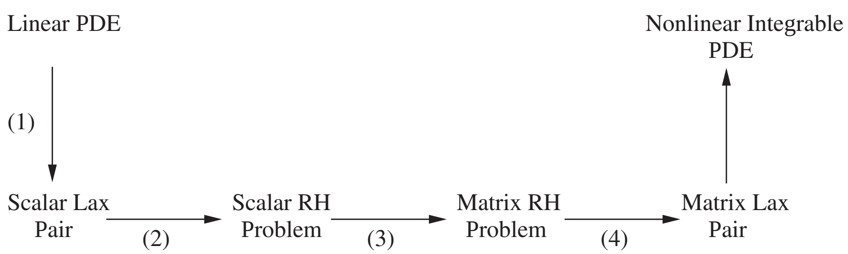


Diagram 15.1

15.1 A Lax Pair Formulation

When replacing $w(k)$ by $iw(k)$ in (1.1), we wee that (1.1) becomes (15.1). Hence, when replacing $w(k)$ by $iw(k)$ in (1.15), (9.12), and (9.14), we obtain the following result.

Proposition 15.1. The function $q(x,t)$ satisfies the linear dispersive evolution PDE (15.1a) if and only if the following two linear eigenvalue equations for the scalar function $\mu(x,t,k)$ are compatible:

$$\frac{\partial \mu}{\partial x} - ik\mu = q(x,t), \tag{15.2a}$$

$$\frac{\partial \mu}{\partial t} + iw(k)\mu = \sum_{j=0}^{n-1} c_j(k)\partial_x^j q(x,t), \quad k \in \mathbb{C}, \tag{15.2b}$$

where the polynomials $\{c_j(k)\}_0^{n-1}$ are defined in terms of $w(k)$ by the identity

$$\sum_{j=0}^{n-1} c_j(k)\partial_x^j = -\frac{w(k)-w(l)}{k-l}\bigg|_{l=-i\partial_x}. \tag{15.3}$$

Furthermore, (15.2) can be rewritten in the form

$$\begin{aligned} &d\left[e^{-ikx+iw(k)t}\mu(x,t,k)\right] \\ &\quad = e^{-ikx+iw(k)t}\left[q(x,t)dx + \sum_{j=0}^{n-1} c_j(k)\partial_x^j q(x,t)dt\right], \quad k \in \mathbb{C}. \end{aligned} \tag{15.4}$$

Proof. Differentiating (15.2a) with respect to t and using (15.2b) we find

$$\frac{\partial^2 \mu}{\partial t \partial x} = ik[-iw\mu + \Sigma] + \frac{\partial q}{\partial t},$$

where $\sum$ denotes the RHS of (15.2b). Similarly, differentiating equation (15.2b) respect to x and using (15.2a) we find

$$\frac{\partial^2 \mu}{\partial x \partial t} = -iw[ik\mu + q] + \frac{\partial}{\partial x}\Sigma.$$

Subtracting the above two equations and using (15.3) we find

$$\frac{\partial^2 \mu}{\partial t \partial x} - \frac{\partial^2 \mu}{\partial x \partial t} = q_t + iw(-i\partial_x)q.$$

□

15.2 A Scalar RH Problem

The Cauchy problem of (15.1a) on the infinite line can be solved in an elementary way through the Fourier transform in the variable x. In what follows we express this result in the form of a scalar RH problem.

Proposition 15.2. Let $q(x,t)$ satisfy (15.1a) in $\{x \in \mathbb{R}, t > 0\}$ and let $q(x,0) = q_0(x)$, $x \in \mathbb{R}$, where $q_0(x)$ has sufficient smoothness, as well as sufficient decay as $|x| \to \infty$. Then $q(x,t)$ is given by

$$q(x,t) = -i \lim_{k\to\infty} [k\mu(x,t,k)], \quad x \in \mathbb{R}, \quad t > 0, \tag{15.5}$$

where $\mu(x,t,k)$ is the solution of the following scalar RH problem in $k \in \mathbb{C}$, for all $x \in \mathbb{R}$, $t > 0$:

$$\mu = \begin{cases} \mu^+, & \operatorname{Im} k \geq 0, \\ \mu^-, & \operatorname{Im} k \leq 0, \end{cases} \tag{15.6a}$$

$$\mu = O\left(\frac{1}{k}\right), \quad k \to \infty, \tag{15.6b}$$

$$\mu^+(x,t,k) - \mu^-(x,t,k) = e^{ikx - iw(k)t}\hat{q}_0(k), \quad k \in \mathbb{R}, \tag{15.6c}$$

where $\hat{q}_0(k)$ is the Fourier transform of $q_0(x)$, i.e.,

$$\hat{q}_0(k) = \int_{-\infty}^{\infty} e^{-ikx} q_0(x)dx, \quad k \in \mathbb{R}. \tag{15.7}$$

Proof. The unique solution of the RH problem (15.6) is given by

$$\mu(x,t,k) = \frac{1}{2i\pi}\int_{-\infty}^{\infty} e^{ilx - iw(l)t}\hat{q}_0(l)\frac{dl}{l-k}, \quad k \in \mathbb{C}\backslash\mathbb{R}, \quad x \in \mathbb{R}, \quad t > 0. \tag{15.8}$$

Equations (15.5) and (15.8) imply

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ilx - iw(l)t}\hat{q}_0(l)dl, \quad x \in \mathbb{R}, \quad t > 0. \tag{15.9}$$

The RHS of this equation involves (x,t) in the exponential form $\exp[ikx - iw(k)t]$, and hence it immediately follows that $q(x,t)$ satisfies (15.1a). Also, evaluating (15.9) at $t = 0$ we find

$$q(x,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ilx}\hat{q}_0(l)dl, \quad x \in \mathbb{R}. \tag{15.10}$$

This equation implies that $q(x,0) = q_0(x)$ provided that one assumes the validity of the classical Fourier transform result. An alternative derivation of this result is presented in Example 6.1. This derivation is based on the spectral analysis of the equation

$$\frac{\partial \mu_0}{\partial x}(k,t) - ik\mu_0(x,t) = q_0(x), \quad k \in \mathbb{C}, \quad x \in \mathbb{R},$$

and since $q_0(x)$ is given, all the steps presented in Example 6.1 of Chapter 6 can be rigorously justified. ∎

15.3 A Matrix RH Problem

It is straightforward to verify that the scalar RH problem satisfying (15.6b) and (15.6c) can be rewritten in the following triangular matrix form (see section I.5.1 of the introduction):

$$M(x,t,k) = \operatorname{diag}(1,1) + O\left(\frac{1}{k}\right), \quad k \to \infty, \tag{15.11a}$$

$$M^+(x,t,k) = M^-(x,t,k)\begin{pmatrix} 1 & e^{ikx-iw(k)t}\hat{q}_0(k) \\ 0 & 1 \end{pmatrix}, \quad k \in \mathbb{R}, \tag{15.11b}$$

where $x \in \mathbb{R}$, $t > 0$, and M is defined in terms of μ by

$$M = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}.$$

The 2×2 jump matrix appearing in the RHS of (15.11b) has unit determinant. In order to obtain a genuine—as opposed to a triangular-matrix RH problem, we replace the above matrix with a full matrix which, however, retains the important property of unit determinant (so that the relevant RH problem has zero index [17]). Hence, we replace the matrix appearing in (15.11b) with the following jump matrix:

$$J(x,t,k) = \begin{pmatrix} 1 & e^{ikx-iw(k)t}\rho_1(k) \\ e^{-ikx+iw(k)t}\rho_2(k) & 1+\rho_1(k)\rho_2(k) \end{pmatrix}, \tag{15.12}$$

where $\rho_1(k)$ and $\rho_2(k)$ are some functions of k. It was indicated in section I.5.1 of the introduction that a 2×2 matrix RH problem with the jump matrix J yields a matrix Lax pair and hence a system of nonlinear PDEs.

15.4 The Dressing Method

The dressing method, in addition to providing an algorithmic way for deriving integrable nonlinear PDEs, is also important for the rigorous treatment of the solution of initial, as well as initial-boundary, value problems. In the latter case, the "jump" of the associated RH problem does not occur across the real line but across a certain curve $\mathcal{L}$ in the complex k-plane. In anticipation of this application (see Chapter 16), we will implement the dressing method, assuming that J is defined for $k \in \mathcal{L}$.

In sections 15.1–15.3 we have implemented steps (1)–(3) of Diagram 15.1 for the general equation (15.1a). However, the implementation of step (4) depends on the particular form of $w(k)$. In what follows, for brevity of presentation we consider the simple case of $w = k^2$.

Proposition 15.3 (the NLS equation). Suppose that the oriented smooth curve $\mathcal{L}$ divides the complex k-plane into the domains D^+ and D^-, where D^+ is to the left of the increasing direction of $\mathcal{L}$. Let $M(x, t, k)$ satisfy the following 2×2 matrix RH problem in the complex k-plane for all $(x, t) \in \Omega \subset \mathbb{R}^2$:

$$M = \begin{cases} M^+, & k \in D^+, \\ M^-, & k \in D^-, \end{cases} \tag{15.13a}$$

$$M = \operatorname{diag}(1, 1) + O\left(\frac{1}{k}\right), \quad k \to \infty, \tag{15.13b}$$

$$M^+(x, t, k) = M^-(x, t, k)e^{-i(kx+2k^2t)\sigma_3} S(k) e^{i(kx+2k^2t)\sigma_3}, \quad k \in \mathcal{L}, \tag{15.13c}$$

where

$$\sigma_3 = \operatorname{diag}(1, -1) \tag{15.14}$$

and $S(k)$ is a 2×2 unimodular matrix with $(S)_{11} = 1$. Assume that the above RH problem has a unique solution which is sufficiently smooth for all $(x, t) \in \Omega$. Then M satisfies the following pair of eigenvalue equations for all $k \in \mathbb{C}$ and $(x, t) \in \Omega$:

$$M_x + ik[\sigma_3, \mu] = QM, \tag{15.15a}$$

$$M_t + 2ik^2[\sigma_3, \mu] = (2kQ - iQ_x\sigma_3 - iQ^2\sigma_3)M, \tag{15.15b}$$

where $[\,,]$ denotes the usual matrix commutator, and the 2×2 off-diagonal matrix Q is defined by

$$Q(x, t) = i \lim_{k \to \infty} [\sigma_3, kM(x, t, k)]. \tag{15.16}$$

Furthermore, the matrix-valued function $Q(x, t)$ satisfies the following nonlinear evolution equation:

$$iQ_t - Q_{xx}\sigma_3 + 2Q^3\sigma_3 = 0. \tag{15.17}$$

Proof. We first note that the jump matrix appearing in the RHS of (15.13c) corresponds to the matrix J defined in (15.12) with $w = k^2$, where for convenience we have replaced k with $-2k$.

The main idea of the dressing method is to construct two linear operators L and N such that (i) LM and NM satisfy the same jump condition as M, and (ii) LM and NM are of $O(1/k)$ as $k \to \infty$. Then, under the *assumption* that the RH problem defined by (15.13) has a unique solution, it follows that

$$LM = 0, \quad NM = 0. \tag{15.18}$$

These equations constitute the Lax pair associated with the above RH problem.

In order to construct L we introduce the operator $\hat{\sigma}_3$ defined by

$$\hat{\sigma}_3 M = [\sigma_3, M]. \tag{15.19}$$

Using the identity

$$e^{\hat{\sigma}_3} A = e^{\sigma_3} A e^{-\sigma_3},$$

it follows that (15.13c) can be rewritten in the form

$$M^+ = M^- e^{-i(kx+2k^2t)\hat{\sigma}_3} S(k). \tag{15.20}$$

This equation immediately implies that M satisfies the equation

$$\{(\partial_x + ik\hat{\sigma}_3)M^+\} = \{(\partial_x + ik\hat{\sigma}_3)M^-\}e^{-i(kx+2k^2t)\hat{\sigma}_3} S(k), \tag{15.21}$$

as well as a similar equation with the operator $\partial_x + ik\hat{\sigma}_3$ replaced by the operator $\partial_t + 2ik^2\hat{\sigma}_3$.

Since M satisfies the boundary condition (15.13b), it follows that $(\partial_x + ik\hat{\sigma}_3)M$ is of $O(1)$ as $k \to \infty$. Thus, in order to construct an operator L such that LM is of $O(1/k)$ as $k \to \infty$ we must subtract $Q(x,t)M$ (note that QM satisfies the same jump condition as M). Thus, we define L by

$$LM \doteqdot M_x + ik[\sigma_3, M] - QM. \tag{15.22}$$

Substituting the asymptotic expansion

$$M(x,t,k) = \operatorname{diag}(1,1) + \frac{M_1(x,t)}{k} + \frac{M_2(x,t)}{k^2} + O\left(\frac{1}{k^3}\right), \quad k \to \infty, \tag{15.23}$$

into (15.22) we find

$$LM = \{i[\sigma_3, M_1] - Q\} + O\left(\frac{1}{k}\right), \quad k \to \infty.$$

Thus, if Q is defined by the equation

$$Q(x,t) = i[\sigma_3, M_1(x,t)], \tag{15.24}$$

which is (15.16), then LM satisfies the following homogeneous RH problem:

$$(LM)^+ = (LM)^- e^{-i(kx+2k^2t)\sigma_3} S(k) e^{i(kx+2k^2t)\sigma_3},$$

$$LM = O\left(\frac{1}{k}\right), \quad k \to \infty.$$

Hence, $LM = 0$; i.e., M and Q satisfy (15.15a).

The operator $(\partial_t + 2ik^2\hat{\sigma}_3)M$ is of order $O(k)$; thus we define N by

$$NM \doteqdot M_t + 2ik^2[\sigma_3, M] - kA(x,t)M - B(x,t)M, \tag{15.25}$$

where A and B are to be determined. Substituting the asymptotic expansion (15.23) into (15.25), we find

$$NM = k\{2i[\sigma_3, M_2] - A\} + \{2i[\sigma_3, M_2] - AM_1 - B\} + O\left(\frac{1}{k}\right).$$

Thus, we define A and B by the equations

$$A = 2i[\sigma_3, M_1], \tag{15.26}$$

and

$$B = 2i[\sigma_3, M_2] - AM_1. \tag{15.27}$$

Comparing (15.24) and (15.26) it follows that

$$A = 2Q. \tag{15.28}$$

Then (15.27) becomes

$$B = -2(QM_1 - i[\sigma_3, M_2]). \tag{15.29}$$

The $O(1/k)$ term in the asymptotic expansion of the equation $LM = 0$ yields

$$M_{1_x} + i[\sigma_3, M_2] = QM_1. \tag{15.30}$$

Comparing this equation with (15.29) it follows that $B = -2M_{1_x}$, i.e.,

$$B = -2\partial_x \left[M_1^{(0)} + M_1^{(D)} \right], \tag{15.31}$$

where the superscripts refer to the off-diagonal and the diagonal parts of the matrix M_1, respectively. Equation (15.24) implies that

$$M_1^{(0)} = \frac{i}{2} Q\sigma_3. \tag{15.32}$$

The diagonal part of (15.30) yields

$$M_{1_x}^{(D)} = QM_1^{(0)} = \frac{i}{2} Q^2\sigma_3. \tag{15.33}$$

Hence, (15.31)–(15.33) imply

$$B = -i(Q_x\sigma_3 + Q^2\sigma_3). \tag{15.34}$$

The equation $NM = 0$, where N is defined by (15.25) and A, B are defined by (15.28), (15.34), is (15.15b).

The compatibility condition of (15.15) yields (15.17). □

Remark 15.1. Denoting the (1-2) and (2-1) components of Q by q and r, respectively, and considering the (1-2) and (2-1) components of (15.17), we find (80) of the introduction. The reduction $r = \lambda\bar{q}$, $\lambda = \pm 1$, yields the celebrated NLS equation (81).

Chapter 16

Nonlinear Integrable PDEs on the Half-Line

Let $q(x,t)$ satisfy the linear PDE (1.1) in the half-line, the initial condition $q(x,0) = q_0(x)$, and a set of appropriate boundary conditions at $x = 0$. *Assume* that there exists a solution with appropriate smoothness and decay. Then, by performing the *simultaneous* spectral analysis of the associated Lax pair, it is possible to express this solution as an integral in the complex k-plane in terms of $\hat{q}_0(k)$ (the Fourier transform of $q_0(x)$) and of $\tilde{g}(k)$ (which involves transforms of all the boundary values); see (1.16). Furthermore, $\hat{q}_0(k)$ and $\tilde{g}(k)$ can be expressed in terms of $\psi(x,k)$ and $\varphi(t,k)$, respectively, where these latter functions are appropriate solutions of the x-part of the Lax pair evaluated at $t = 0$ and of the t-part of the Lax pair evaluated at $x = 0$ (for the particular case of the linear PDE (71), ψ and φ satisfy (86) and (87)). The above integral representation involves the transforms of *all* the boundary values $\{\partial_x^j q(0,t)\}_0^{n-1}$; thus in order for this representation to provide the solution of a given initial-boundary value problem, it is first necessary to eliminate the unknown boundary values. This can be achieved by employing the associated global relation.

The situation for nonlinear integrable PDEs is *conceptually* similar. Before summarizing the main steps needed for the analysis of a nonlinear PDE, we first introduce some notations.

Notations

- $\Omega = \{0 < x < \infty, 0 < t < T\}$, T positive finite constant.
- $\mu(x,t,k)$, $(x,t) \in \Omega$, $k \in \mathbb{C}$, will denote a 2×2 matrix-valued sectionally holomorphic solution of the associated Lax pair.
- $\Psi(x,k)$, $0 < x < \infty$, will denote an appropriate solution of the x-part of the Lax pair evaluated at $t = 0$.
- $\Phi(t,k)$, $0 < t < T$, will denote an appropriate solution of the t-part of the Lax pair evaluated at $x = 0$.
- $q_0(x) = q(x,0)$, $0 < x < \infty$, will denote the given initial condition, which will be assumed to belong to the Schwartz class, i.e., $q_0(x) \in S(\mathbb{R}^+)$.

- The set of functions $\{a(k), b(k), A(k), B(k)\}$ will be referred to as the *spectral functions*.

Let $q(x,t)$ satisfy an integrable nonlinear PDE with spatial derivatives of order up to n. Suppose that this PDE possesses a 2×2 matrix Lax pair formulation. Then, the analysis of this PDE on the half-line involves the following three steps.

1. Assume that $q(x, t)$ exists.

- *Direct Problem.* Define $\mu(x,t,k)$ for all $k \in \mathbb{C}$ in terms of $q(x,t)$ (via a Volterra integral equation).
- *Inverse Problem.* Define $q(x,t)$ in terms of the spectral functions (via a 2×2 matrix Reimann–Hilbert (RH) problem), where

$$\{a(k), b(k)\} \text{ are defined in terms of } q_0(x) (\text{via } \Psi(x,k))$$

and

$$\{A(k), B(k)\} \text{ are defined in terms of } \{\partial_x^j q(0,t)\}_0^{n-1} (\text{via } \Phi(t,k)).$$

- *Global Relation.* Derive the relation coupling $\{a(k), b(k), A(k), B(k)\}$.

2. Assume that the spectral functions satisfy the global relation.

- Define $\{a(k), b(k)\}$ in terms of $q_0(x)$ (via $\Psi(x,k)$) and analyze their properties.
- Define $\{A(k), B(k)\}$ in terms of $\{g_j(t)\}_0^{n-1}$ (via $\Phi(t,k)$) and analyze their properties.
- Define $q(x,t)$ in terms of $\{a(k), b(k), A(k), B(k)\}$ via an RH problem and prove that $q(x,t)$ solves the given nonlinear PDE; furthermore prove that

$$q(x,0) = q_0(x); \quad \partial_x^j q(0,t) = g_j(t), \quad j = 0, 1, \ldots, n-1.$$

3. The Analysis of the Global Relation.

Given a subset of the functions $\{\partial_x^j q(0,t)\}_0^{n-1}$, determine $\{A(k), B(k)\}$ by employing the global relation.

In sections 16.1 and 16.2 we will implement the first two steps for the nonlinear Schrödinger (NLS) and to the following equations respectively: the sine-Gordon (sG), the modified Korteweg–de Vries (KdV), and the KdV equations. The third step for these equations will be discussed in Chapters 17 and 18.

16.1 The NLS Equation

The NLS equation (81) admits the following Lax pair formulation:

$$\mu_x + ik[\sigma_3, \mu] = Q(x,t)\mu, \tag{16.1a}$$

$$\mu_t + 2ik^2[\sigma_3, \mu] = \tilde{Q}(x,t,k)\mu, \tag{16.1b}$$

where $\sigma_3 = \text{diag}(1, -1)$,

$$Q(x,t) = \begin{pmatrix} 0 & q(x,t) \\ \lambda \bar{q}(x,t) & 0 \end{pmatrix}, \qquad \tilde{Q}(x,t,k) = 2kQ - iQ_x\sigma_3 - i\lambda|q|^2\sigma_3. \tag{16.2}$$

16.1.1. Assume That $q(x,t)$ Exists

Assuming that $q(x,t)$ is known, equations (16.1) can be considered as *two* equations for the *single* function $\mu(x,t,k)$. These two equations are compatible provided that $q(x,t)$ satisfies the NLS equation (81). Indeed, computing μ_{xt} from (16.1a) and μ_{tx} from (16.1b), it can be shown that $\mu_{xt} = \mu_{tx}$, provided that $q(x,t)$ solves the NLS.

Let $\hat{\sigma}_3$ denote the commutator with respect to σ_3; then if A is a 2×2 matrix, the expression $(\exp \hat{\sigma}_3)A$ takes a simple form:

$$\hat{\sigma}_3 A = [\sigma_3, A], \qquad e^{\hat{\sigma}_3} A = e^{\sigma_3} A e^{-\sigma_3}. \tag{16.3}$$

Equations (16.1) can be rewritten as the following *single* equation:

$$d\left(e^{i(kx+2k^2t)\hat{\sigma}_3}\mu(x,t,k)\right) = W(x,t,k), \tag{16.4}$$

where the exact 1-form W is defined by

$$W = e^{i(kx+2k^2t)\hat{\sigma}_3}(Q\mu dx + \tilde{Q}\mu dt). \tag{16.5}$$

The advantage of this formalism is that it provides a straightforward way for obtaining an expression for $\mu(x,t,k)$ using the fundamental theorem of calculus; see Chapter 10.

16.1.1.1 The Direct Problem

We will construct a sectionally holomorphic solution μ in terms of $q(x,t)$. Assuming that the function $q(x,t)$ has sufficient smoothness and decay, a solution of (16.4) is given by

$$\mu_j(x,t,k) = I + \int_{(x_j,t_j)}^{(x,t)} e^{-i(kx+2k^2t)\hat{\sigma}_3} W(\xi,\tau,k), \tag{16.6}$$

where I is the 2×2 identity matrix, (x_j, t_j) is an arbitrary point in the domain $0 < \xi < \infty$, $0 < \tau < T$, and the integral is over a piecewise smooth curve from (x_j,t_j) to (x,t). Since the 1-form W is exact, μ_j is independent of the path of integration. The analyticity properties of μ_j with respect to k depend on the choice of (x_j,t_j). It was shown in [22] that for a polygonal domain there exists a canonical way of choosing the points (x_j,t_j), namely they are the corners of the associated polygon. Thus, we define three different solutions μ_1, μ_2, μ_3, corresponding to $(0,T)$, $(0,0)$, (∞,t); see Figure 10.1. Also we choose the particular contours shown in Figure 10.1.

This choice implies the following inequalities on the contours:

$$\begin{aligned} \mu_1 &: \quad \xi - x \le 0, \qquad \tau - t \ge 0, \\ \mu_2 &: \quad \xi - x \le 0, \qquad \tau - t \le 0, \\ \mu_3 &: \quad \xi - x \ge 0. \end{aligned}$$

Using the identity

$$e^{-\alpha\hat{\sigma}_3}A = e^{-\alpha\sigma_3}Ae^{\alpha\sigma_3} = \begin{pmatrix} e^{-\alpha} & 0 \\ 0 & e^{\alpha} \end{pmatrix}\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} e^{\alpha} & 0 \\ 0 & e^{-\alpha} \end{pmatrix}$$

$$= \begin{pmatrix} A_{11} & e^{-2\alpha}A_{12} \\ e^{2\alpha}A_{21} & A_{22} \end{pmatrix},$$

it follows that the second column of the matrix equation (16.6) involves $\exp[2ik(\xi - x) + 4ik^2(\tau - t)]$. Using the above inequalities it follows that this exponential is bounded in the following domains of the complex k-plane:

$$\mu_1 \quad : \quad \{\text{Im}\, k \le 0 \cap \text{Im}\, k^2 \ge 0\},$$

$$\mu_2 \quad : \quad \{\text{Im}\, k \le 0 \cap \text{Im}\, k^2 \le 0\},$$

$$\mu_3 \quad : \quad \{\text{Im}\, k \ge 0\}.$$

Thus the second column vectors of μ_1, μ_2, and μ_3 are bounded and analytic for $\arg k$ in $(\pi, 3\pi/2)$, $(3\pi/2, 2\pi)$, and $(0, \pi)$, respectively. We will denote these vectors with superscripts (3), (4), and (12) to indicate that they are bounded and analytic in the third quadrant, fourth quadrant, and the upper half of the complex k-plane, respectively. Similar conditions are valid for the first column vectors; thus

$$\mu_1(x,t,k) = (\mu_1^{(2)}, \mu_1^{(3)}), \quad \mu_2(x,t,k) = (\mu_2^{(1)}, \mu_2^{(4)}), \quad \mu_3(x,t,k) = (\mu_3^{(34)}, \mu_3^{(12)}). \tag{16.7}$$

For T finite, the functions μ_1 and μ_2 are entire functions of k.

Equation (16.6) and integration by parts imply that in the domains where $\{\mu_j\}$ are bounded, the following estimate is valid:

$$\mu_j(x,t,k) = I + O\left(\frac{1}{k}\right), \quad k \to \infty, \; j = 1, 2, 3. \tag{16.8}$$

The 2×2 matrix μ consisting of the collection of the functions $\{\mu_j\}_1^3$ provides the solution of the direct problem. Indeed, in each quadrant of the complex k-plane there exist two column vectors which are bounded and analytic. In the first, second, third, and fourth quadrants these vectors are, respectively,

$$\left(\mu_2^{(1)}, \mu_3^{(12)}\right), \left(\mu_1^{(2)}, \mu_3^{(12)}\right), \left(\mu_3^{(34)}, \mu_1^{(3)}\right), \left(\mu_3^{(34)}, \mu_2^{(4)}\right).$$

16.1.1.2 The Inverse Problem

Equations (16.6) and (16.7) define μ in terms of q. Now, by exploiting the analytic dependence of μ on k, we will express μ in terms of the spectral functions. This will be achieved by formulating an RH problem. In order to formulate this RH problem, we need to compute the "jumps" across the real and the imaginary k-axis of the vectors appearing in (16.7). It turns out that the relevant jump matrices can be uniquely defined in terms of the following 2×2 matrix-valued functions:

$$s(k) = \mu_3(0, 0, k) \quad \text{and} \quad S(k) = [e^{2ik^2T\hat{\sigma}_3}\mu_2(0, T, k)]^{-1}. \tag{16.9}$$

This is a direct consequence of the fact that any two solutions of (16.6) are simply related. For example,

$$\mu_3(x, t, k) = \mu_2(x, t, k)e^{-i(kx+2k^2t)\hat{\sigma}_3}\mu_3(0, 0, k). \tag{16.10}$$

Similarly,

$$\mu_1(x, t, k) = \mu_2(x, t, k)e^{-i(kx+2k^2t))\hat{\sigma}_3}[e^{2ik^2T\hat{\sigma}_3}\mu_2(0, T, k)]^{-1}. \tag{16.11}$$

The functions $s(k)$ and $S(k)$ follow from the evaluations at $x = 0$ and $t = T$, respectively, of the function $\mu_3(x, 0, k)$ and of $\mu_2(0, t, k)$. These functions satisfy the following linear integral equations:

$$\mu_3(x, 0, k) = I + \int_\infty^x e^{ik(\xi-x)\hat{\sigma}_3}(Q\mu_3)(\xi, 0, k)\, d\xi, \tag{16.12}$$

$$\mu_2(0, t, k) = I + \int_0^t e^{2ik^2(\tau-t)\hat{\sigma}_3}(\tilde{Q}\mu_2)(0, \tau, k)\, d\tau. \tag{16.13}$$

The matrix-valued function $S(k)$ can be alternatively defined through the equation

$$S(k) = \mu_1(0, 0, k);$$

this definition is more convenient in the case when $T = \infty$.

The Spectral Functions

The entries of the matrices $s(k)$ and $S(k)$ are *not* independent. This is a consequence not only of the fact that each of the matrices $\{\mu_j\}_1^3$ has a unit determinant, but also of the fact that Q satisfies the symmetry condition $(Q)_{21} = \lambda(Q)_{12}$. Indeed, the fact that Q and $\tilde{Q}$ are traceless, together with (16.8), implies $\det \mu_j(x, t, k) = 1$, $j = 1, 2, 3$. Thus

$$\det s(k) = \det S(k) = 1. \tag{16.14}$$

The symmetry properties of Q and $\tilde{Q}$ imply

$$(\mu(x, t, k))_{11} = \overline{(\mu(x, t, \bar{k}))_{22}}, \quad (\mu(x, t, k))_{21} = \lambda\overline{\mu(x, t, \bar{k})_{12}},$$

and thus

$$s_{11}(k) = \overline{s_{22}(\bar{k})}, \quad s_{21}(k) = \lambda\overline{s_{12}(\bar{k})}, \quad S_{11}(k) = \overline{S_{22}(\bar{k})}, \quad S_{21}(k) = \lambda\overline{S_{12}(\bar{k})}.$$

These equations justify the following notation for s and S:

$$s(k) = \begin{pmatrix} \overline{a(\bar{k})} & b(k) \\ \lambda\overline{b(\bar{k})} & a(k) \end{pmatrix}, \qquad S(k) = \begin{pmatrix} \overline{A(\bar{k})} & B(k) \\ \lambda\overline{B(\bar{k})} & A(k) \end{pmatrix}. \tag{16.15}$$

The definitions of $\mu_j(0,t,k)$, $j = 1, 2$, and of $\mu_2(x, 0, k)$ imply that the spectral functions have larger domains of boundedness, namely,

$$\mu_1(0,t,k) = \left(\mu_1^{(24)}(0,t,k), \mu_1^{(13)}(0,t,k)\right), \tag{16.16a}$$

$$\mu_2(0,t,k) = \left(\mu_2^{(13)}(0,t,k), \mu_2^{(24)}(0,t,k)\right), \tag{16.16b}$$

$$\mu_2(x,0,k) = \left(\mu_2^{(12)}(x,0,k), \mu_2^{(34)}(x,0,k)\right). \tag{16.16c}$$

The definitions of $s(k)$, $S(k)$ and the notations (16.15) imply

$$\begin{pmatrix} b(k) \\ a(k) \end{pmatrix} = \mu_3^{(12)}(0,0,k), \qquad \begin{pmatrix} -e^{-4ik^2T}B(k) \\ \overline{A(\bar{k})} \end{pmatrix} = \mu_2^{(24)}(0,T,k), \tag{16.17}$$

where the vectors $\mu_3^{(12)}(x,0,k)$ and $\mu_2^{(24)}(0,t,k)$ satisfy the following ODEs:

$$\begin{gathered} \partial_x \mu_3^{(12)}(x,0,k) + 2ik \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mu_3^{(12)}(x,0,k) = Q(x,0)\mu_3^{(12)}(x,0,k), \\ 0 \le \arg k \le \pi, \ 0 < x < \infty, \\ \lim_{x\to\infty} \mu_3^{(12)}(x,0,k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \end{gathered} \tag{16.18}$$

$$\begin{gathered} \partial_t \mu_2^{(24)}(0,t,k) + 4ik^2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mu_2^{(24)}(0,t,k) = \tilde{Q}(0,t,k)\mu_2^{(24)}(0,t,k), \\ \arg k \in [\pi/2, \pi] \cup [3\pi/2, 2\pi], \ 0 < t < T, \\ \mu_2^{(24)}(0,0,k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{gathered} \tag{16.19}$$

The above definitions imply the following properties.

$a(k)$ and $b(k)$

$a(k)$, $b(k)$ are defined and analytic for $\arg k \in (0, \pi)$.

$|a(k)|^2 - \lambda|b(k)|^2 = 1, k \in \mathbb{R}$.

$$a(k) = 1 + O\left(\frac{1}{k}\right), \ b(k) = O\left(\frac{1}{k}\right), \quad k \to \infty. \tag{16.20}$$

$A(k)$ and $B(k)$

$A(k)$, $B(k)$ are entire functions bounded for $\arg k \in [0, \frac{\pi}{2}] \cup [\pi, \frac{3\pi}{2}]$. If $T = \infty$, the functions $A(k)$ and $B(k)$ are defined and analytic in the quadrants $\arg k \in (0, \frac{\pi}{2}) \cup (\pi, \frac{3\pi}{2})$.

$A(k)\overline{A(\bar{k})} - \lambda B(k)\overline{B(\bar{k})} = 1, \ k \in \mathbb{C}$ ($k \in \mathbb{R} \cup i\mathbb{R}$, if $T = \infty$).

$$A(k) = 1 + O\left(\frac{1}{k}\right) + O\left(\frac{e^{4ik^2T}}{k}\right), \quad B(k) = O\left(\frac{1}{k}\right) + O\left(\frac{e^{4ik^2T}}{k}\right), \quad k \to \infty. \tag{16.21}$$

All of the above properties, except for the property that $B(k)$ is bounded for $\arg k \in [0, \pi/2] \cup [\pi, 3\pi/2]$, follow from the analyticity and boundedness of $\mu_3(x, 0, k)$, $\mu_2(0, t, k)$, from the conditions of unit determinant, and from the large k asymptotics of these eigenfunctions. Regarding $B(k)$ we note that $B(k) = B(T, k)$, where

$$B(t, k) = -\exp(4ik^2t)\left(\mu_2^{(24)}(0, t, k)\right)_1$$

and the subscript (1) denotes the first component of the vector $\mu_2^{(24)}$. Equations (16.19) imply a linear Volterra integral equation for the vector $\exp(4ik^2t)\mu_2^{(24)}(0, t, k)$, from which it immediately follows that $B(t, k)$ is an entire function of k bounded for $\arg k \in [0, \pi/2] \cup [\pi, 3\pi/2]$.

The RH Problem

Equations (16.10) and (16.11) can be rewritten in the form expressing the jump condition of a 2×2 RH problem. This involves tedious but straightforward algebraic manipulations, which will be presented in section 16.2. The final form is

$$M_-(x, t, k) = M_+(x, t, k)J(x, t, k), \quad k \in \mathbb{R} \cup i\mathbb{R}, \tag{16.22}$$

where the matrices M_-, M_+, and J are defined as follows:

$$\begin{aligned}
M_+ &= \left(\frac{\mu_2^{(1)}}{a(k)}, \mu_3^{(12)}\right), \ \arg k \in \left[0, \frac{\pi}{2}\right];\\
M_- &= \left(\frac{\mu_1^{(2)}}{d(k)}, \mu_3^{(12)}\right), \ \arg k \in \left[\frac{\pi}{2}, \pi\right];\\
M_+ &= \left(\mu_3^{(34)}, \frac{\mu_1^{(3)}}{\overline{d(\bar{k})}}\right), \ \arg k \in \left[\pi, \frac{3\pi}{2}\right];\\
M_- &= \left(\mu_3^{(34)}, \frac{\mu_2^{(4)}}{\overline{a(\bar{k})}}\right), \ \arg k \in \left[\frac{3\pi}{2}, 2\pi\right];
\end{aligned} \tag{16.23}$$

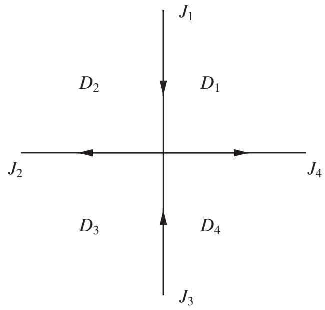


Figure 16.1. *The contour for the RH problem for the NLS.*

$$d(k) = a(k)\overline{A(\bar{k})} - \lambda b(k)\overline{B(\bar{k})}; \tag{16.24}$$

$$J(x,t,k) = \begin{cases} J_4, & \arg k = 0, \\ J_1, & \arg k = \frac{\pi}{2}, \\ J_2 = J_3 J_4^{-1} J_1, & \arg k = \pi, \\ J_3, & \arg k = \frac{3\pi}{2}; \end{cases} \tag{16.25}$$

with

$$J_1 = \begin{pmatrix} 1 & 0 \\ \Gamma(k)e^{2i\theta} & 1 \end{pmatrix}, \quad J_4 = \begin{pmatrix} 1 & -\gamma(k)e^{-2i\theta} \\ \lambda\bar{\gamma}(k)e^{2i\theta} & 1-\lambda|\gamma(k)|^2 \end{pmatrix},$$
$$J_3 = \begin{pmatrix} 1 & -\lambda\overline{\Gamma(\bar{k})}e^{-2i\theta} \\ 0 & 1 \end{pmatrix}; \tag{16.26}$$

$$\theta(x,t,k) = kx + 2k^2t; \quad \gamma(k) = \frac{b(k)}{\bar{a}(k)}, \quad k \in \mathbb{R}; \quad \Gamma(k) = \frac{\lambda\overline{B(\bar{k})}}{a(k)d(k)}, \quad k \in \mathbb{R}^- \cup i\mathbb{R}^+. \tag{16.27}$$

The contour for this RH problem is depicted in Figure 16.1.

The matrix $M(x,t,k)$ defined by (16.23) is in general a meromorphic function of k in $\mathbb{C} \setminus \{\mathbb{R} \cup i\mathbb{R}\}$. The possible poles of M are generated by the zeros of $a(k)$ and $d(k)$ and by the complex conjugate of these zeros.

We will make the following *assumptions* regarding these zeros.

1. $a(k)$ has n simple zeros $\{k_j\}_1^n$, $n = n_1 + n_2$, where $\arg k_j \in (0, \frac{\pi}{2})$, $j = 1, \dots, n_1$; $\arg k_j \in (\frac{\pi}{2}, \pi)$, $j = n_1+1, \dots, n_1+n_2$.

2. $d(k)$ has Λ simple zeros $\{\lambda_j\}_1^\Lambda$, where $\arg \lambda_j \in (\frac{\pi}{2}, \pi)$, $j = 1, \dots, \Lambda$.

3. None of the zeros of $a(k)$ for $\arg k \in (\frac{\pi}{2}, \pi)$ coincides with a zero for $d(k)$.

In order to evaluate the associated residues we introduce the following notation:

$[A]_1$ (resp., $[A]_2$) denote the first (resp., second) column of A and $\dot{a}(k) = \frac{da}{dk}$.

The following formulae are valid:

$$\operatorname*{Res}_{k_j} [M(x,t,k)]_1 = \frac{1}{\dot{a}(k_j)b(k_j)} e^{2i\theta(k_j)}[M(x,t,k_j]_2, \quad j = 1, \dots, n_1, \tag{16.28a}$$

$$\operatorname*{Res}_{\bar{k}_j} [M(x,t,k)]_2 = \frac{\lambda}{\dot{\bar{a}}(k_j)\bar{b}(k_j)} e^{-2i\theta(\bar{k}_j)}[M(x,t,\bar{k}_j)]_1, \quad j = 1, \dots, n_1, \tag{16.28b}$$

$$\operatorname*{Res}_{\lambda_j} [M(x,t,k)]_1 = \frac{\lambda \overline{B(\bar{\lambda}_j)}}{a(\lambda_j)\dot{d}(\lambda_j)} e^{2i\theta(\lambda_j)}[M(x,t,\lambda_j)]_2, \quad j = 1, \dots, \Lambda, \tag{16.28c}$$

$$\operatorname*{Res}_{\bar{\lambda}_j} [M(x,t,k)]_2 = \frac{B(\bar{\lambda}_j)}{\bar{a}(\lambda_j)\dot{\bar{d}}(\lambda_j)} e^{-2i\theta(\bar{\lambda}_j)}[M(x,t,\bar{\lambda}_j)]_1, \quad j = 1, \dots, \Lambda, \tag{16.28d}$$

where

$$\theta(k_j) = k_j x + 2k_j^2 t. \tag{16.29}$$

In order to derive (16.28a) we note that the second column of (16.10) is

$$\mu_3^{(12)} = a\mu_2^{(4)} + b\mu_2^{(1)}e^{-2i\theta}.$$

Recalling that μ_2 is an entire function and evaluating this equation at $k = k_j$, $j = 1, \dots, n_1$, we find

$$\mu_3^{(12)}(k_j) = b(k_j)e^{-2i\theta(k_j)}\mu_2^{(1)}(k_j),$$

where for simplicity of notation we have suppressed the x, t dependence. Thus

$$\operatorname*{Res}_{k_j} [M]_1 = \frac{\mu_2^{(1)}(k_j)}{\dot{a}(k_j)} = \frac{e^{2i\theta(k_j)}\mu_3^{(12)}(k_j)}{\dot{a}(k_j)b(k_j)},$$

which is (16.28a), since $\mu_3^{(12)}(k_j) = [M]_2(k_j)$.

In order to derive (16.28c) we note that the first column of the equation $M_- = M_+ J_1$ yields

$$a\mu_1^{(2)} = d\mu_2^{(1)} + \lambda \bar{B} e^{2i\theta}\mu_3^{(12)}.$$

Evaluating this equation at $k = \lambda_j$ (each term has an analytic continuation into the second quadrant) and using

$$\operatorname*{Res}_{\lambda_j} [M]_1 = \frac{\mu_1^{(2)}(\lambda_j)}{\dot{d}(\lambda_j)}, \quad [M]_2 = \mu_3^{(12)},$$

we find (16.28c).

Remarks 16.1. 1. The column $[\mu_3(x,0,k_j)]_2$ is a nontrivial vector solution of (16.1a) evaluated at $t=0$. Therefore, $a(k)$ and $b(k)$ cannot have common zeros, and hence $b(k_j)\neq 0$. Similar arguments together with the third assumption above imply that $B(\bar{\lambda}_j)\neq 0$.

2. By extending $q_0(x)$ to the whole axis, $q_0(x)=0$, $x<0$, we can identify the set $\{k_j\}_1^n$ of zeros of $a(k)$ as the discrete spectrum of the Dirac operator associated with the NLS equation considered on the whole axis. If $\lambda=1$, this operator is self-adjoint. This implies that the set $\{k_j\}_1^n$ is empty when $\lambda=1$. However, we do not have a similar argument for the function $d(k)$. Therefore, in order to ensure the solvability of the RH problem in the defocusing case we shall *assume* that $d(k)$ has no zeros if $\lambda=1$. The asymptotic analysis of the RH problem (16.22) in the defocusing case suggests that this problem has *no* solution if $d(k)$ has zeros. Thus, we *conjecture* that solitons do not exist for $\lambda=1$.

$q(x,t)$ in Terms of the Spectral Functions

In order to rewrite the jump condition (16.22) in a more convenient form, we introduce the matrix $\tilde{J}$ such that $J=I+\tilde{J}$, i.e., we define the matrices $\{\tilde{J}_j\}_1^4$ as follows:

$$\tilde{J}_1=\begin{pmatrix}0 & 0\\ \Gamma e^{2i\theta} & 0\end{pmatrix},\ \tilde{J}_2=\begin{pmatrix}-\lambda(|\gamma|^2+|\Gamma|^2)+\Gamma\gamma+\bar{\Gamma}\bar{\gamma} & (\gamma-\lambda\bar{\Gamma})e^{-2i\theta}\\ (\Gamma-\lambda\bar{\gamma})e^{2i\theta} & 0\end{pmatrix},$$

$$\tilde{J}_3=\begin{pmatrix}0 & -\lambda\bar{\Gamma}e^{-2i\theta}\\ 0 & 0\end{pmatrix},\quad \tilde{J}_4=\begin{pmatrix}0 & -\gamma e^{-2i\theta}\\ \lambda\bar{\gamma}e^{2i\theta} & -\lambda|\gamma|^2\end{pmatrix}. \tag{16.30}$$

Replacing J by $I+\tilde{J}$ in (16.22), we find

$$M_+-M_-=-M_+\tilde{J},\quad k\in\mathbb{R}\cup i\mathbb{R}. \tag{16.31a}$$

This equation, together with the estimate

$$M=I+O\left(\frac{1}{k}\right),\quad k\to\infty, \tag{16.31b}$$

implies

$$M(x,t,k)=I-\frac{1}{2i\pi}\int_{\mathbb{R}\cup i\mathbb{R}}M_+(x,t,l)\tilde{J}(x,t,l)\frac{dl}{l-k},\quad k\in\mathbb{C}\backslash\mathbb{R}\cup i\mathbb{R}. \tag{16.32}$$

The large-k asymptotics of this equation implies

$$M(x,t,k)=I+\frac{M_1(x,t)}{k}+O\left(\frac{1}{k^2}\right),\quad k\to\infty, \tag{16.33a}$$

where

$$M_1(x,t)=\frac{1}{2i\pi}\int_{\mathbb{R}\cup i\mathbb{R}}(M_+\tilde{J})(x,t,k)dk. \tag{16.33b}$$

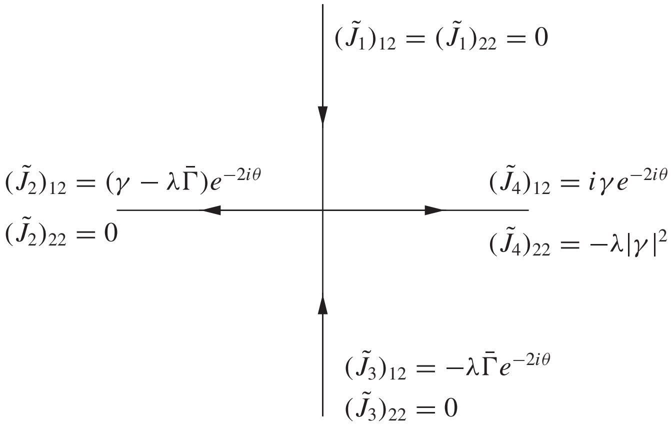


Figure 16.2. *The entries of* $\{\tilde{J}\}_1^4$.

Substituting the RHS of (16.33a) in the x-part of the Lax pair, i.e., in (16.1a), we find

$$Q = i[\sigma_3, M_1];$$

hence,

$$q(x,t) = 2i(M_1(x,t))_{12} = \frac{1}{\pi}\int_{\mathbb{R}\cup i\mathbb{R}} (M_+\tilde{J})(x,t,k)dk.$$

Using

$$(M_+\tilde{J})_{12} = (M_+)_{11}\tilde{J}_{12} + (M_+)_{12}\tilde{J}_{22},$$

as well as the expressions for $\tilde{J}_{12}$ and $\tilde{J}_{22}$ defined by (16.30), we find the following expression for q (see Figure 16.2):

$$q(x,t) = -\frac{1}{\pi}\left\{\int_{-\infty}^{\infty}(M_+)_{11}(x,t,k)\gamma(k)e^{-2i\theta}dk + \lambda\int_0^{\infty}(M_+)_{12}(x,t,k)|\gamma(k)|^2 dk\right.$$
$$\left. + \lambda\int_{\partial D_3}(M_+)_{11}(x,t,k)\overline{\Gamma(\bar{k})}e^{-2i\theta}dk\right\}, \quad (x,t)\in\Omega, \tag{16.34}$$

where ∂D_3 denotes the oriented boundary of the third quadrant of the complex k-plane.

Equation (16.34) with $\lambda = 1$ is (92).

16.1.1.3 The Global Relation

Evaluating (16.10) at $x = 0$, $t = T$ and using (16.9), we find

$$\mu_3(0,T,k) = \left(e^{-2ik^2T\hat{\sigma}_3}S^{-1}(k)\right)\left(e^{-2ik^2T\hat{\sigma}_3}s(k)\right).$$

Multiplying this equation by $\exp[2ik^2T\hat{\sigma}_3]$ and using the definition of $\mu_3(x,t,k)$, the above equation becomes

$$-I + S(k)^{-1}s(k) + e^{2ik^2T\hat{\sigma}_3}\int_0^{\infty} e^{ik\xi\hat{\sigma}_3}(Q\mu_3)(\xi,T,k)d\xi = 0. \tag{16.35}$$

The (1-2) component of this equation yields the following global relation:

$$B(k)a(k) - A(k)b(k) = e^{4ik^2T}c^+(k), \quad \arg k \in [0, \pi], \tag{16.36}$$

where

$$c^+(k) = \int_0^\infty e^{2ik\xi} q(\xi, T)(\mu_3)_{22}(\xi, T, k)dk. \tag{16.37}$$

Remark 16.2 (the linear limit). Letting $q = \varepsilon v + O(\varepsilon^2)$ it follows that as $\varepsilon \to 0$, v satisfies the linear PDE (72). The solution of this linear PDE can be expressed in the form (83); replacing k by $-2k$ in this equation we find that for $(x, t) \in \Omega$, $v(x, t)$ is given by

$$v(x, t) = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-2i\theta}\hat{v}_0(k)dk + \frac{1}{\pi}\int_{\partial D_3} e^{-2i\theta}\tilde{v}(k)dk, \tag{16.38a}$$

where θ is as defined in (16.27), ∂D_3 denotes the oriented boundary of the third quadrant of the complex k-plane, and the functions $\hat{v}_0$, $\tilde{v}$ are defined as follows (see (8) and (84)):

$$\hat{v}_0(k) = \int_0^\infty e^{2ikx} v_0(x)dx, \quad \operatorname{Im} k \geq 0, \tag{16.38b}$$

$$\tilde{v}(k) = \int_0^T e^{4ik^2 s}[iv_x(0, s) + 2kv(0, s)]ds, \quad k \in \mathbb{C}, \tag{16.38c}$$

with $v_0(x) = v(x, 0)$. Furthermore, the global relation becomes

$$\hat{v}_0(k) - \tilde{v}(k) = e^{4ik^2T}c^+(k), \quad \operatorname{Im} k \geq 0. \tag{16.38d}$$

We will now show that the formulae for the solution of the NLS reduce in the linear limit to (16.38). Indeed, the definitions of $\{\mu_j\}_1^3$ imply that if $q \sim \varepsilon v$, then $\{\mu_j\}_1^3 \sim I$. Hence the definitions of $s(k)$ and $S(k)$, i.e., equations (16.9), imply that as $\varepsilon \to 0$,

$$s(k) = I - \varepsilon \int_0^\infty e^{ik\xi\hat{\sigma}_3} U(\xi, 0)d\xi + O(\varepsilon^2),$$

$$S^{-1}(k) = I + \varepsilon \int_0^T e^{2ik^2\tau\hat{\sigma}_3}\tilde{U}(0, \tau)d\tau + O(\varepsilon^2),$$

where U and $\tilde{U}$ denote the expressions obtained from Q and $\tilde{Q}$ after replacing q with v. Using the notations (16.15) for s and S, as well as the definitions for Q and $\tilde{Q}$, i.e., (16.2), we find that as $\varepsilon \to 0$,

$$a(k) = 1 + O(\varepsilon), \; A(k) + 1 + O(\varepsilon), \; b(k) = -\varepsilon\hat{v}_0(k) + O(\varepsilon^2), \; B(k) = -\varepsilon\tilde{v}(k) + O(\varepsilon^2). \tag{16.39}$$

Hence, the definitions of $\gamma(k)$ and $\Gamma(k)$, as $\varepsilon \to 0$, imply

$$\gamma(k) = -\varepsilon\hat{v}_0(k) + O(\varepsilon^2), \quad \lambda\overline{\Gamma(\bar{k})} = \varepsilon\tilde{v}(k) + O(\varepsilon^2).$$

Substituting these expressions, as well as the estimate $M = I + O(\varepsilon)$, in (16.34) and comparing the resulting equation with (16.38a), we find

$$q(x, t) = \varepsilon v(x, t) + O(\varepsilon^2).$$

Furthermore, the $O(\varepsilon)$ term of (16.36) yields (16.38d).

Remark 16.3 (asymptotics). The formalism presented earlier has the following two apparent disadvantages: (a) It is based on the a priori *assumption* of existence, and (b) it yields a formula for $q(x,t)$ which (via the spectral functions) depends on $q_0(x)$ as well as on *both* $q(0,t)$ and $q_x(0,t)$, while only *one* of these functions (or their combination) can be prescribed as a boundary condition. However, regarding (a) we note that the assumption of existence can be *eliminated* by employing standard PDE techniques. For example, for the case that $q(0,t)$ is prescribed, global well-posedness is established in [98] (for the KdV see [99], [100], [101]). Regarding (b) we note that the most important contribution of the analytical formalism of integrable evolution PDEs is the use of the exact formulae for the derivation of certain *asymptotic formulae*. For example, in the case of the initial-value problem of the NLS, the relevant analytical formalism yields explicit formulae for both the long-time asymptotics [102] and for the semiclassical limit [103] (for the KdV see [104], [69]). The derivation of these important formulae is based on the fact that the solution $q(x,t)$ can be expressed through the solution of a matrix RH problem which involves a jump with explicit (x,t) dependence of the form $\exp[2i\theta]$. Equation (16.34) shows that the new transform method applied to nonlinear integrable PDEs on the half-line, just like the case of the initial-value problems, also yields $q(x,t)$ through the solution of a matrix RH problem with explicit (x,t) dependence of the form $\exp[2i\theta]$; see equations (16.22)–(16.27) (the fact that the jump occurs on a more complicated contour does *not* impose any serious difficulties). Hence, it is still possible using this RH problem to obtain *explicit asymptotic formulae* for both the long-time asymptotics [64] and the semiclassical limit [71]. The only difference is that, while these asymptotic formulae involve the spectral functions explicitly, some of the functions, namely $A(k)$ and $B(k)$, *cannot* in general be determined explicitly in terms of the given boundary conditions. This difficulty will be addressed in Chapter 18. Here we only note that there exist some particular boundary conditions for which it *is* possible to determine $A(k)$ and $B(k)$ explicitly; such boundary conditions, which will be referred to as *linearizable*, will be discussed in Chapter 17.

16.1.2. Assume That the Spectral Functions Satisfy the Global Relation

The analysis of section 16.1.1 motivates the following definitions for the spectral functions.

Definition 16.1 (the spectral functions $a(k)$ and $b(k)$). Given $q_0(x) \in S(\mathbb{R}^+)$, we define the map

$$\mathbb{S} : \{q_0(x)\} \mapsto \{a(k), b(k)\} \tag{16.40}$$

as follows:

$$a(k) = \varphi_2(0,k), \qquad b(k) = \varphi_1(0,k), \qquad \operatorname{Im} k \geq 0, \tag{16.41}$$

where the vector $\varphi(x,k) = (\varphi_1, \varphi_2)$ is the unique solution of

$$\begin{aligned} &\varphi_{1_x} + 2ik\varphi_1 = q_0(x)\varphi_2, \\ &\varphi_{2_x} = \lambda\bar{q}_0(x)\varphi_1, \quad \operatorname{Im} k \geq 0, \quad 0 < x < \infty, \\ &\lim_{x\to\infty} \varphi = (0,1). \end{aligned} \tag{16.42}$$

The functions a and b are well defined, since equations (16.42) are equivalent to the vector Volterra linear integral equation,

$$\varphi_1(x,k) = -\int_x^{\infty} e^{-2ik(x-y)} q_0(y)\varphi_2(y,k)dy, \tag{16.43a}$$

$$\varphi_2(x,k) = 1 - \lambda\int_x^{\infty} \bar{q}_0(y)\varphi_1(y,k)dy, \mathrm{Im}\, k \geq 0. \tag{16.43b}$$

Proposition 16.1 (properties of $a(k)$ and $b(k)$). The spectral functions $a(k)$ and $b(k)$ defined by (16.41) have the following properties:

(i) $a(k)$ and $b(k)$ are analytic for $\mathrm{Im}\, k > 0$ and continuous and bounded for $\mathrm{Im}\, k \geq 0$.

(ii) $a(k) = 1 + O(\frac{1}{k}),\ b(k) = O(\frac{1}{k}),\ k \to \infty$.

(iii) $|a(k)|^2 - \lambda|b(k)|^2 = 1,\ k \in \mathbb{R}$.

(iv) The map

$$\mathbb{Q} : \{a(k), b(k)\} \mapsto \{q_0(k)\},$$

which is inverse to $\mathbb{S}$, is defined as follows:

$$q_0(x) = 2i \lim_{k\to\infty} (kM^{(x)}(x,k))_{12}, \tag{16.44}$$

where $M^{(x)}(x,k)$ is the unique solution of the following RH problem:

- $$M^{(x)}(x,k) = \begin{cases} M_-^{(x)}(x,k), & \mathrm{Im}\, k \leq 0, \\ M_+^{(x)}(x,k), & \mathrm{Im}\, k \geq 0, \end{cases} \tag{16.45a}$$

  is a sectionally meromorphic function.

- $$M_-^{(x)}(x,k) = M_+^{(x)}(x,k)J^{(x)}(x,k), \quad k \in \mathbb{R}, \tag{16.45b}$$

  where

  $$J^{(x)}(x,k) = \begin{pmatrix} 1 & -\dfrac{b(k)}{\bar{a}(k)}e^{-2ikx} \\ \frac{\lambda\bar{b}(k)}{a(k)}e^{2ikx} & \frac{1}{|a|^2} \end{pmatrix}. \tag{16.45c}$$

- $$M^{(x)}(x,k) = I + O(\frac{1}{k}), \quad k \to \infty. \tag{16.45d}$$

- We assume that if $\lambda = -1$, $a(k)$ can have at most n simple zeros $\{k_j\}_1^n$, $n = n_1 + n_2$, where $\arg k_j \in (0, \frac{\pi}{2})$, $j = 1, \ldots, n_1$; $\arg k_j \in (\frac{\pi}{2}, \pi)$, $j = n_1 + 1, \ldots, n_1 + n_2$.

- If $\lambda = -1$, the first column of $M_+^{(x)}$ can have simple poles at $k = k_j$, $j = 1, \ldots, n$, and the second column of $M_-^{(x)}$ can have simple poles at $k = \bar{k}_j$, where $\{k_j\}_1^n$ are the simple zeros of $a(k)$, Im $k > 0$. The associated residues are given by

$$\operatorname*{Res}_{k_j} [M^{(x)}(x,k)]_1 = \frac{e^{2ik_jx}}{\dot{a}(k_j)b(k_j)}[M^{(x)}(x,k_j)]_2,$$
$$\operatorname*{Res}_{\bar{k}_j} [M^{(x)}(x,k)]_2 = \frac{\lambda e^{-2i\bar{k}_jx}}{\overline{\dot{a}(k_j)b(k_j)}}[M^{(x)}(x,\bar{k}_j)]_1. \tag{16.45e}$$

(v) $\mathbb{S}^{-1} = \mathbb{Q}$.

Proof. Properties (i)–(iii) follow from Definition 16.1. In order to derive properties (iv) and (v) we define the vector function $\psi(x,k) = (\psi_1, \psi_2)$ as the unique solution of the following problem:

$$\psi_{1,x} = q_0(x)\psi_2,$$
$$\psi_{2,x} - 2ik\psi_2 = \lambda\bar{q}_0(x)\psi_1, \quad 0 < x < \infty, \quad k \in \mathbb{C},$$
$$\psi(0,k) = (1,0).$$

Note that the vector ψ satisfies the vector linear Volterra equation

$$\psi_1(x,k) = 1 + \int_0^x q_0(y)\psi_2(y,k)dy,$$
$$\psi_2(x,k) = \lambda\int_0^x e^{2ik(x-y)}\bar{q}_0(y)\psi_1(y,k)dy, k \in \mathbb{C}.$$

We introduce the notations

$$\varphi^*(x,k) = \left(\overline{\varphi_2(x,\bar{k})}, \lambda\overline{\varphi_1(x,\bar{k})}\right), \quad \psi^*(x,k) = \left(\overline{\psi_2(x,\bar{k})}, \lambda\overline{\psi_1(x,\bar{k})}\right).$$

Let $\mu_3(x,k)$ and $\mu_2(x,k)$ be defined by

$$\mu_3(x,k) = \left(\varphi^*(x,k), \varphi(x,k)\right), \quad \mu_2(x,k) = \left(\psi(x,k), \lambda\psi^*(x,k)\right).$$

These functions satisfy the matrix equation

$$\mu_x + ik[\sigma_3, \mu] = \begin{pmatrix} 0 & q_0 \\ \lambda\bar{q}_0 & 0 \end{pmatrix}\mu. \tag{16.46}$$

This in turn implies that the above vectors are simply related, namely

$$\left(\varphi^*(x,k), \varphi(x,k)\right) = \left(\psi(x,k), \lambda\psi^*(x,k)\right)e^{-ikx\hat{\sigma}_3}s(k), \quad k \in \mathbb{R}. \tag{16.47}$$

Let

$$M_-^{(x)} = \left(\varphi^*, \frac{\lambda\psi^*}{\bar{a}(\bar{k})}\right), \quad \operatorname{Im} k \le 0,$$
$$M_+^{(x)} = \left(\frac{\psi}{a(k)}, \varphi\right), \quad \operatorname{Im} k \ge 0. \tag{16.48}$$

Equation (16.47) can be rewritten as

$$M_-^{(x)}(x,k) = M_+^{(x)}(x,k)J^{(x)}(x,k), \quad k \in \mathbb{R}, \tag{16.49}$$

where $J^{(x)}(x,k)$ is the jump matrix defined by (16.45c). Furthermore, $M^{(x)}$ satisfies the RH problem defined in (16.45). Indeed, we need only prove the residue conditions at the possible simple zeros $\{k_j\}_1^n$ of $a(k)$. To this end we note that in virtue of (16.47) the following equation is valid:

$$\varphi = b(k)e^{-2ikx}\psi + a(k)\lambda\psi^*. \tag{16.50}$$

The function ψ and hence the function ψ^* are entire functions of k. Therefore, we can evaluate (16.50) at $k = k_j$. This yields the relation

$$\varphi(x,k_j) = \psi(x,k_j)b(k_j)e^{-2ik_jx},$$

or, taking into account the definition (16.48) of the function $M^{(x)}(x,k)$, we find the following residue condition:

$$\operatorname{Res}_{k_j}[M^{(x)}(x,k_j)]_1 = \frac{e^{2ik_jx}}{\dot{a}(k_j)b(k_j)}[M^{(x)}(x,k_j)]_2.$$

The residue condition at $k = \bar{k}_j$ can be derived similarly.

Substituting the asymptotic expansion

$$M^{(x)}(x,k) = I + \frac{m_1(x)}{k} + O\left(\frac{1}{k^2}\right), \quad k \to \infty,$$

into (16.46) we find

$$q_0(x) = 2i\Big(m_1(x)\Big)_{12} = 2i \lim_{k\to\infty}\Big(kM^{(x)}(x,k)\Big)_{12}. \tag{16.51}$$

Our next task is to show that this relation defines the map

$$\mathbb{Q} : \{a(k), b(k)\} \mapsto \{q_0(x)\},$$

which is inverse to the map

$$\mathbb{S} : \{q_0(x)\} \mapsto \{a(k), b(k)\}.$$

In more detail this problem can be formulated as follows. Given $\{a(k), b(k)\}$, construct the jump matrix $J^{(x)}(x,k)$ according to (16.45c) and define the RH problem by (16.45). Let $q_0(x)$ be the function defined by (16.51) in terms of the solution $M^{(x)}(x,k)$ of this RH problem. Denote by $\{a_0(k), b_0(k)\}$ the spectral data corresponding to $q_0(x)$. We must show that

$$a_0(k) = a(k) \quad \text{and} \quad b_0(k) = b(k). \tag{16.52}$$

Using the standard arguments of the dressing method (see Chapter 15), it is straightforward to prove that $M^{(x)}(x,k)$ satisfies (16.46) with the function $q_0(x)$ defined by (16.51). This means in particular that the matrix solution $\mu_3(x,k)$, $k \in \mathbb{R}$, corresponding to the function $q_0(x)$ is given by the equation

$$\mu_3(x,k) = M_+^{(x)}(x,k)e^{-ikx\hat{\sigma}_3}C_+(k), \quad k \in \mathbb{R}, \tag{16.53}$$

for some matrix $C_+(k)$. This matrix does not depend on x and hence can be evaluated by letting $x \to \infty$ in (16.53).

It follows from the theory of the inverse scattering problem for the Dirac equation (16.46) (see, e.g., [125], or from the direct use of the nonlinear steepest descent method, [67]) that under the usual assumptions on the RH data $\{a(k), b(k)\}$ the following estimate holds:

$$M_+^{(x)}(x,k) = \begin{pmatrix} 1 & 0 \\ -\frac{\lambda\bar{b}(k)}{a(k)}e^{2ikx} & 1 \end{pmatrix} + o(1), \quad x \to \infty, \quad k \in \mathbb{R}.$$

Since $\mu_3 \to I$ as $x \to \infty$, it follows that

$$C_+(k) = \begin{pmatrix} 1 & 0 \\ \frac{\lambda\bar{b}(k)}{a(k)} & 1 \end{pmatrix}. \tag{16.54}$$

Equations (16.53) and (16.54) imply that the scattering data

$$s_0(k) = \begin{pmatrix} \bar{a_0}(k) & b_0(k) \\ \lambda\bar{b_0}(k) & a_0(k) \end{pmatrix} = \mu_3(0,k)$$

corresponding to the function $q_0(x)$ defined in (16.51) are given by the equation

$$s_0(k) = M_+^{(x)}(0,k)\begin{pmatrix} 1 & 0 \\ \frac{\lambda\bar{b}(k)}{a(k)} & 1 \end{pmatrix}.$$

If $x = 0$ (in fact, for all $x \leq 0$), the above RH problem can be solved explicitly. Indeed,

$$J^{(x)}(0,k) = \begin{pmatrix} 1 & -\frac{b(k)}{\bar{a}(k)} \\ \frac{\lambda\bar{b}(k)}{a(k)} & \frac{1}{|a|^2} \end{pmatrix} = \begin{pmatrix} a(k) & -b(k) \\ 0 & \frac{1}{a(k)} \end{pmatrix}\begin{pmatrix} \bar{a}(k) & 0 \\ \lambda\bar{b}(k) & \frac{1}{\bar{a}(k)} \end{pmatrix}.$$

Using the fact that the residue conditions are satisfied, this implies

$$M_+^{(x)}(0,k) = \begin{pmatrix} \frac{1}{a(k)} & b(k) \\ 0 & a(k) \end{pmatrix},$$

and hence

$$s_0(k) = \begin{pmatrix} \frac{1}{a(k)} & b(k) \\ 0 & a(k) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{\lambda\bar{b}(k)}{a(k)} & 1 \end{pmatrix} = \begin{pmatrix} \bar{a}(k) & b(k) \\ \lambda\bar{b}(k) & a(k) \end{pmatrix} = s(k),$$

i.e., (16.52) follows. □

Remark 16.4. The properties of $a(k)$ and $b(k)$ imply that $a(k)$ can be expressed in terms of $b(k)$. Indeed, if $a(k) \neq 0$, for $\operatorname{Im} k \geq 0$, then

$$a(k) = \exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{\infty} \ln(1+\lambda|b(k')|^2)\frac{dk'}{k'-k}\right\}, \quad \operatorname{Im} k > 0. \tag{16.55}$$

Also, the upper half plane analyticity of $b(k)$ implies that

$$b(k) = \int_0^\infty \hat{b}(s)e^{iks}ds,$$

where $\hat{b}(s)$ is a complex-valued function of Schwartz type on $\mathbb{R}^+$ (if the same behavior is assumed for $q_0(x)$). Thus, if $a(k) \neq 0$, the maps $\mathbb{S}$ and $\mathbb{Q}$ define the bijection

$$q_0(x) \longleftrightarrow b(k)\,. \tag{16.56}$$

If $\lambda = -1$ and $a(k)$ has zeros, the equation for $a(k)$ must be replaced by

$$a(k) = \prod_{j=1}^{n}\frac{k-k_j}{k-\bar{k}_j}\exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{\infty} \ln(1+\lambda|b(k')|^2)\frac{dk'}{k'-k}\right\}, \quad \operatorname{Im} k > 0,$$

and a discrete component $\{k_j\}$ must be added to the RHS of (16.56).

Definition 16.2 (the spectral functions $A(k)$ and $B(k)$). Let

$$\tilde{Q}(t,k) = 2k\begin{pmatrix} 0 & g_0(t) \\ \lambda\overline{g_0}(t) & 0 \end{pmatrix} - i\begin{pmatrix} 0 & g_1(t) \\ \lambda\overline{g_1}(t) & 0 \end{pmatrix}\sigma_3 - i\lambda|g_0(t)|^2\sigma_3, \quad \lambda = \pm 1. \tag{16.57}$$

Let $g_0(t)$ and $g_1(t)$ be smooth functions. The map

$$\widetilde{\mathbb{S}} : \{g_0(t), g_1(t)\} \to \{A(k), B(k)\} \tag{16.58}$$

is defined as follows:

$$\begin{pmatrix} -e^{-4ik^2T}B(k) \\ \overline{A(\bar{k})} \end{pmatrix} = \Phi(T,k), \quad k \in \mathbb{C}, \tag{16.59}$$

where the vector $\Phi(t,k) = (\Phi_1, \Phi_2)$ is the unique solution of

$$\begin{aligned} &\Phi_{1_t} + 4ik^2\Phi_1 = \tilde{Q}_{11}\Phi_1 + \tilde{Q}_{12}\Phi_2, \\ &\Phi_{2_t} = \tilde{Q}_{21}\Phi_1 + \tilde{Q}_{22}\Phi_2, \quad 0 < t < T, \quad k \in \mathbb{C}, \\ &\Phi(0,k) = (0,1). \end{aligned} \tag{16.60}$$

The functions $A(k)$ and $B(k)$ are well defined, since equations (16.60) are equivalent to the linear vector Volterra integral equation

$$\Phi_1(t,k) = \int_0^t e^{-4ik^2(t-\tau)}(\tilde{Q}_{11}\Phi_1 + \tilde{Q}_{12}\Phi_2)(\tau,k)d\tau, \quad k \in \mathbb{C}, \tag{16.61a}$$

$$\Phi_2(t,k) = 1 + \int_0^t (\tilde{Q}_{21}\Phi_1 + \tilde{Q}_{22}\Phi_2)(\tau,k)d\tau, \quad k \in \mathbb{C}. \tag{16.61b}$$

If $T = \infty$, we assume that the functions $g_0(t)$ and $g_1(t)$ belong to $S(\mathbb{R}_+)$, and we use an alternative definition of the spectral functions $A(k)$ and $B(k)$ based on the solution $\mu_1(0,t,k)$; namely we let

$$\begin{pmatrix} B(k) \\ A(k) \end{pmatrix} = \tilde{\Phi}(0,k), \quad \operatorname{Im} k^2 \geq 0,$$

where the vector $\tilde{\Phi}(t,k) = (\tilde{\Phi}_1, \tilde{\Phi}_2)$ is the unique solution of

$$\begin{gathered} \tilde{\Phi}_{1_t} + 4ik^2\tilde{\Phi}_1 = \tilde{Q}_{11}\tilde{\Phi}_1 + \tilde{Q}_{12}\tilde{\Phi}_2, \\ \tilde{\Phi}_{2_t} = \tilde{Q}_{21}\tilde{\Phi}_1 + \tilde{Q}_{22}\tilde{\Phi}_2, \quad t > 0, \quad \operatorname{Im} k^2 \geq 0, \\ \lim_{t\to\infty} \tilde{\Phi}(t,k) = (0,1). \end{gathered}$$

In the case of $T < \infty$, this definition is equivalent to (16.59). The functions $\tilde{\Phi}_1(t,k)$ and $\tilde{\Phi}_2(t,k)$ satisfy the linear vector Volterra integral equation

$$\tilde{\Phi}_1(t,k) = -\int_t^\infty e^{-4ik^2(t-\tau)}(\tilde{Q}_{11}\tilde{\Phi}_1 + \tilde{Q}_{12}\tilde{\Phi}_2)(\tau,k)d\tau,$$

$$\tilde{\Phi}_2(t,k) = 1 - \int_t^\infty (\tilde{Q}_{21}\tilde{\Phi}_1 + \tilde{Q}_{22}\tilde{\Phi}_2)(\tau,k)d\tau.$$

Therefore, in the case of $T = \infty$ the spectral functions $A(k)$ and $B(k)$ are well defined and analytic only for $\arg k \in [0, \frac{\pi}{2}] \cup [\pi, \frac{3\pi}{2}]$.

Proposition 16.2 (properties of $A(k)$ and $B(k)$). The spectral functions $A(k)$ and $B(k)$ defined above have the following properties:

(i) $A(k)$, $B(k)$ are entire functions bounded for k in the first and third quadrants, i.e., $\arg k \in [0, \frac{\pi}{2}] \cup [\pi, \frac{3\pi}{2}]$. If $T = \infty$, the functions $A(k)$ and $B(k)$ are defined only for k in these quadrants.

(ii) $A(k) = 1 + O\left(\frac{1}{k}\right) + O\left(\frac{e^{4ik^2T}}{k}\right)$, $B(k) = O\left(\frac{1}{k}\right) + O\left(\frac{e^{4ik^2T}}{k}\right)$, $k \to \infty$.

(iii) $A(k)\overline{A(\bar{k})} - \lambda B(k)\overline{B(\bar{k})} = 1$, $k \in \mathbb{C}$ ($k \in \mathbb{R} \cup i\mathbb{R}$, if $T = \infty$).

(iv) The map

$$\widetilde{\mathbb{Q}} : \{A(k), B(k)\} \mapsto \{g_0(t), g_1(t)\},$$

which is inverse to $\widetilde{\mathbb{S}}$, is defined as follows:

$$\begin{aligned} g_0(t) &= 2i \lim_{k\to\infty} \left(kM^{(t)}(t,k)\right)_{12}, \\ g_1(t) &= \lim_{k\to\infty} \left[4\left(k^2 M^{(t)}(t,k)\right)_{12} + 2ig_0(t)k\left(M^{(t)}(t,k)\right)_{22}\right], \end{aligned} \tag{16.62}$$

where $M^{(t)}(t,k)$ is the unique solution of the following RH problem:

-
$$M^{(t)}(t,k) = \begin{cases} M_+^{(t)}(t,k), & \arg k \in [0, \frac{\pi}{2}] \cup [\pi, \frac{3\pi}{2}], \\ M_-^{(t)}(t,k), & \arg k \in [\frac{\pi}{2}, \pi] \cup [\frac{3\pi}{2}, 2\pi], \end{cases} \tag{16.63a}$$

  is a sectionally meromorphic function.

-
$$M_-^{(t)}(t,k) = M_+^{(t)}(t,k) J^{(t)}(t,k), \quad k \in \mathbb{R} \cup i\mathbb{R}, \tag{16.63b}$$

  where

$$J^{(t)}(t,k) = \begin{pmatrix} 1 & -\frac{B(k)}{\overline{A(\bar{k})}} e^{-4ik^2t} \\ \frac{\lambda \overline{B(\bar{k})}}{A(k)} e^{4ik^2t} & \frac{1}{A(k)\overline{A(\bar{k})}} \end{pmatrix}. \tag{16.63c}$$

-
$$M^{(t)}(t,k) = I + O\left(\frac{1}{k}\right), \quad k \to \infty. \tag{16.63d}$$

- We assume that $A(k)$ can have at most N simple zeros $\{K_j\}_1^N$, $\arg K_j \in (0, \frac{\pi}{2}) \cup (\pi, \frac{3\pi}{2})$. The first column of $M_+^{(t)}(t,k)$ can have simple poles at $k = K_j$, $j = 1, \ldots, N$, and the second column of $M_-^{(t)}(t,k)$ can have simple poles at

$k = \bar{K}_j$, where $\{K_j\}^N$ are the simple zeros of $A(k)$, $\arg k \in (0, \frac{\pi}{2}) \cup (\pi, \frac{3\pi}{2})$. The associated residues are given by

$$\operatorname*{Res}_{K_j} [M^{(t)}(t,k)]_1 = \frac{\exp[4iK_j^2 t]}{\dot{A}(K_j)B(K_j)}[M^{(t)}(t,K_j)]_2, \qquad j = 1,\ldots,N,$$

$$\operatorname*{Res}_{\bar{K}_j} [M^{(t)}(t,k)]_2 = \frac{\lambda \exp[-4i\bar{K}_j^2 t]}{\overline{\dot{A}(\bar{K}_j)B(\bar{K}_j)}}[M^{(t)}(t,\bar{K}_j)]_1, \quad j = 1,\ldots,N. \tag{16.63e}$$

(v) $\widetilde{\mathbb{S}}^{-1} = \widetilde{\mathbb{Q}}$.

Proof. Properties (i)–(iii) follow from Definition 16.2. In order to derive properties (iv) and (v) we define the vector function $\Psi(x,k) = (\Psi_1, \Psi_2)$ as the unique solution of the following problem:

$$\Psi_{1_t} = \tilde{Q}_{11}\Psi_1 + \tilde{Q}_{12}\Psi_2,$$

$$\Psi_{2_t} - 4ik^2\Psi_2 = \tilde{Q}_{21}\Psi_1 + \tilde{Q}_{22}\Psi_2, \qquad 0 < t < T, \quad k \in \mathbb{C},$$

$$\Psi(T,k) = (1,0),$$

where $\tilde{Q}(t,k)$ is as defined by (16.57). The vector Ψ satisfies the linear vector Volterra equation

$$\Psi_1(t,k) = 1 + \int_T^t (\tilde{Q}_{11}\Psi_1 + \tilde{Q}_{12}\Psi_2)(\tau,k)d\tau,$$

$$\Psi_2(t,k) = \int_T^t e^{4ik^2(t-\tau)}(\tilde{Q}_{21}\Psi_1 + \tilde{Q}_{22}\Psi_2)(\tau,k)d\tau.$$

We introduce the notations

$$\Phi^*(t,k) = \left(\overline{\Phi_2(t,\bar{k})}, \lambda\overline{\Phi_1(t,\bar{k})}\right), \qquad \Psi^*(t,k) = \left(\overline{\Psi_2(t,\bar{k})}, \lambda\overline{\Psi_1(t,\bar{k})}\right).$$

Let μ_1 and μ_2 be defined by

$$\mu_1(t,k) = \left(\Psi(t,k), \lambda\Psi^*(t,k)\right), \quad \mu_2(t,k) = \left(\Phi^*(t,k), \Phi(t,k)\right).$$

These functions satisfy the matrix equation

$$\mu_t + 2ik^2[\sigma_3, \mu] = \tilde{Q}(t,k)\mu. \tag{16.64}$$

This in turn implies that

$$\left(\Phi^*(t,k), \Phi(t,k)\right) = \left(\Psi(t,k), \lambda\Psi^*(t,k)\right)e^{-2ik^2t\hat{\sigma}_3}S(k), \quad k \in \mathbb{R} \cup i\mathbb{R}. \tag{16.65}$$

Let

$$M_-^{(t)} = \left(\Phi^*, \frac{\lambda\Psi^*}{\overline{A(\bar{k})}}\right), \quad \arg k \in [\pi/2, \pi] \cup [3\pi/2, 2\pi],$$

$$M_+^{(t)} = \left(\frac{\Psi}{A(k)}, \Phi\right), \quad \arg k \in [0, \pi/2] \cup [\pi, 3\pi/2]. \tag{16.66}$$

Equation (16.65) can be rewritten as

$$M_-^{(t)}(t,k) = M_+^{(t)}(t,k) J^{(t)}(t,k), \quad k \in \mathbb{R} \cup i\mathbb{R}, \tag{16.67}$$

where $J^{(t)}(t,k)$ is the jump matrix defined in (16.63c). Furthermore, $M^{(t)}$ satisfies the RH problem defined in (16.63). Indeed, as in the x-case, we need only prove the residue conditions at the possible simple zeros $\{K_j\}_1^N$ of $A(k)$. The proof is the same as in the case of the function $M^{(x)}(x,k)$.

Substituting the asymptotic expansion

$$M^{(t)}(t,k) = I + \frac{m_1(t)}{k} + \frac{m_2(t)}{k^2} + O\left(\frac{1}{k^3}\right), \quad k \to \infty,$$

into (16.64) we find the relations

$$g_0(t) = 2i\Big(m_1(t)\Big)_{12} = 2i \lim_{k\to\infty} \Big(kM^{(t)}(t,k)\Big)_{12}, \tag{16.68}$$

$$\begin{aligned} g_1(t) &= 4\Big(m_2(t)\Big)_{12} + 2ig_0(t)\Big(m_1(t)\Big)_{22} \\ &= \lim_{k\to\infty} \left\{4\Big(k^2 M^{(t)}(t,k)\Big)_{12} + 2ig_0(t)k\Big(M^{(t)}(t,k)\Big)_{22}\right\}. \end{aligned} \tag{16.69}$$

We will show that these relations define the map

$$\widetilde{\mathbb{Q}} : \{A(k), B(k)\} \mapsto \{g_0(t), g_1(t)\},$$

which is inverse to the map

$$\widetilde{\mathbb{S}} : \{g_0(t), g_1(t)\} \mapsto \{A(k), B(k)\}.$$

Similarly with the x-case, we have to prove that

$$A_0(k) = A(k) \quad \text{and} \quad B_0(k) = B(k), \tag{16.70}$$

where the LHS is the spectral data corresponding to $g_0(t)$ and $g_1(t)$. We follow precisely the same procedure as the one used for the x-problem: By employing the dressing method it follows that if $M^{(t)}(t,k)$ is the solution of the RH problem, then it satisfies (16.64) with the functions $g_0(t)$and $g_1(t)$ defined by (16.68) and (16.69). This means, in particular, that the matrix solution $\mu_1(t,k)$, with $k \in \mathbb{C}$ (we assume that $T < \infty$) corresponding to the functions $g_0(t)$ and $g_1(t)$, is given by the equation

$$\mu_1(t,k) = M_+^{(t)}(t,k) e^{-2ik^2 t\hat{\sigma}_3} D_+(k), \quad k \in \mathbb{C}, \tag{16.71}$$

for some matrix $D_+(k)$. This matrix does not depend on t and hence can be evaluated by letting $t = T$ in (16.71).

For all t, the jump matrix $J^{(t)}(t,k)$ can be factorized as

$$J^{(t)}(t,k) = \begin{pmatrix} 1 & 0 \\ \frac{\lambda \overline{B(\bar{k})}}{A(k)} e^{4ik^2t} & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{B(k)}{\overline{A(\bar{k})}} e^{-4ik^2t} \\ 0 & 1 \end{pmatrix}. \tag{16.72}$$

Using the asymptotic form of $A(k)$ and $B(k)$ as $k \to \infty$, it follows that

$$\frac{\lambda \overline{B(\bar{k})}}{A(k)} e^{4ik^2T} \to 0, \quad k \to \infty, \quad \arg k \in \left[0, \frac{\pi}{2}\right] \cup \left[\pi, \frac{3\pi}{2}\right], \tag{16.73}$$

and

$$\frac{B(k)}{\overline{A(\bar{k})}} e^{-4ik^2T} \to 0, \quad k \to \infty, \quad \arg k \in \left[\frac{\pi}{2}, \pi\right] \cup \left[\frac{3\pi}{2}, 2\pi\right]. \tag{16.74}$$

Also, taking into account that

$$A(k)\overline{A(\bar{k})} - \lambda B(k)\overline{B(\bar{k})} = 1, \quad k \in \mathbb{C},$$

it follows that if K_j is a zero of $A(k)$, then

$$\operatorname*{Res}_{K_j} \begin{pmatrix} 1 \\ -\frac{\lambda \overline{B(\bar{k})}}{A(k)} e^{4ik^2T} \end{pmatrix} = -\frac{\lambda \overline{B(\bar{K}_j)}}{\dot{A}(K_j)} e^{4iK_j^2T} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\dot{A}(K_j)B(K_j)} e^{4iK_j^2T} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Similarly, at $k = \bar{K}_j$,

$$\operatorname*{Res}_{\bar{K}_j} \begin{pmatrix} -\frac{B(k)}{\overline{A(\bar{k})}} e^{-4ik^2T} \\ 1 \end{pmatrix} = \frac{1}{\overline{\dot{A}(K_j)}\,\overline{B}(K_j)} e^{-4i\bar{K}_j^2T} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

These equations, together with (16.72) and the estimates (16.73), (16.74), imply that for $t = T$ the RH problem defined in (16.63) can be solved explicitly as follows:

$$M_+^{(t)}(T,k) = \begin{pmatrix} 1 & 0 \\ -\frac{\lambda \bar{B}(\bar{k})}{A(k)} e^{4ik^2T} & 1 \end{pmatrix}.$$

Thus

$$D_+(k) = \begin{pmatrix} 1 & 0 \\ \frac{\lambda \bar{B}(\bar{k})}{A(k)} & 1 \end{pmatrix}.$$

If $T = \infty$, the factorization (16.72) fails, however, the methodology of the steepest descent method can still be applied; see Appendix A of [50]. □

Remark 16.5. The properties of $A(k)$ and $B(k)$ imply that $A(k)$ can be expressed in terms of $B(k)$. Indeed, if $A(k) \neq 0$, then

$$A(k) = \prod_{j=1}^{N} \frac{k - K_j}{k - \bar{K}_j} \exp\left\{ \frac{1}{2i\pi} \int_{\mathcal{L}} \ln(1 + \lambda B(k')\overline{B(\bar{k}')}) \frac{dk'}{k' - k} \right\},$$

for $\arg k \in (0, \frac{\pi}{2}) \cup (\pi, \frac{3\pi}{2})$, where the contour $\mathcal{L}$ is the union of the real and the imaginary axes with the orientation shown in Figure 16.1. Also,

$$B(\pm k) = \int_0^\infty \hat{B}_\pm(s) e^{ik^2 s} ds, \quad \arg k \in \left[0, \frac{\pi}{2}\right].$$

Thus, the maps $\widetilde{\mathbb{S}}$ and $\widetilde{\mathbb{Q}}$ define the bijection

$$\{g_0(t), g_1(t)\} \longleftrightarrow \{B(k), K_1, \ldots, K_N, N < \infty\}.$$

Remark 16.6. If $T = \infty$, the functions $g_0(t)$ and $g_1(t)$ are assumed to belong to $S(\mathbb{R}_+)$, and the global relation takes the form

$$a(k)B(k) - b(k)A(k) = 0, \quad \arg k \in \left[0, \frac{\pi}{2}\right]. \tag{16.75}$$

Theorem 16.1 (see [50]). Given $q_0(x) \in S(\mathbb{R}^+)$ define the spectral functions $a(k)$ and $b(k)$ by Definition 16.1. Suppose that there exist smooth functions $g_0(t)$ and $g_1(t)$ satisfying $g_0(0) = q_0(0)$, $g_1(0) = q_0(0)$, such that the functions $\{A(k), B(k)\}$, which are defined in terms of $\{g_0(t), g_1(t)\}$ by Definition 16.2, satisfy the global condition (16.36). Assume that

(i) If $\lambda = -1$, $a(k)$ has at most n simple zeros $\{k_j\}_1^n$, $n = n_1 + n_2$, where $\arg k_j \in (0, \frac{\pi}{2})$, $j = 1, \ldots, n_1$; $\arg k_j \in (\frac{\pi}{2}, \pi)$, $j = n_1 + 1, \ldots, n_1 + n_2$.

(ii) If $\lambda = -1$, the function $d(k)$ defined in terms of the spectral functions by (16.24) has at most Λ simple zeros $\{\lambda_j\}_1^\Lambda$, where

$$\arg \lambda_j \in \left(\frac{\pi}{2}, \pi\right), \quad j = 1, \ldots, \Lambda.$$

If $\lambda = 1$, the function $d(k)$ has no zeros in the second quadrant.

(iii) None of the zeros of $a(k)$ for $\arg k \in (\frac{\pi}{2}, \pi)$ coincides with a zero of $d(k)$.

Define $M(x, t, k)$ as the solution of the following 2×2 matrix RH problem:

- M is sectionally meromorphic in $k \in \mathbb{C} \setminus \{\mathbb{R} \cup i\mathbb{R}\}$.

- The first column of M can have simple poles at k_j, $j = 1, \ldots, n_1$, and λ_j, $j = 1, \ldots, \Lambda$; the second column of M can have simple poles at $\overline{k_j}$, $j = 1, \ldots, n_1$, and $\overline{\lambda_j}$, $j = 1, \ldots, \Lambda$. The associated residues satisfy the relations in (16.28).

- M satisfies the jump condition

$$M_-(x,t,k) = M_+(x,t,k)J(x,t,k), \quad k \in \mathbb{R} \cup i\mathbb{R}, \tag{16.76a}$$

where M is M_- for $\arg k \in [\frac{\pi}{2}, \pi] \cup [\frac{3\pi}{2}, 2\pi]$, M is M_+ for $\arg k \in [0, \frac{\pi}{2}] \cup [\pi, \frac{3\pi}{2}]$, and J is defined in terms of $\{a, b, A, B\}$ by (16.24)–(16.27); see Figure 16.1.

- At ∞,

$$M(x,t,k) = I + O\left(\frac{1}{k}\right), \quad k \to \infty. \tag{16.76b}$$

Then $M(x,t,k)$ exists and is unique.
Define $q(x,t)$ in terms of $M(x,t,k)$ by

$$q(x,t) = 2i \lim_{k\to\infty} \Big(kM(x,t,k)\Big)_{12}, \tag{16.77}$$

which is equivalent to the formula (16.34). Then $q(x,t)$ solves the NLS equation (81). Furthermore,

$$q(x,0) = q_0(x), \quad q(0,t) = g_0(t), \quad q_x(0,t) = g_1(t).$$

Proof. If $\lambda = 1$, then according to Remark 16.1 the function $a(k) \neq 0$ for $\operatorname{Im} k > 0$; also by the assumption that $d(k) \neq 0$ for $\arg k \in (\frac{\pi}{2}, \pi)$. In this case the unique solvability of the RH problem is a consequence of the existence of a "vanishing lemma"; i.e., the RH obtained from the above RH by replacing (16.76b) with $M = O(\frac{1}{k})$, $k \to \infty$, has only the trivial solution. The vanishing lemma can be established using the symmetry properties of J; see [64].

If $\lambda = -1$, $a(k)$ and $d(k)$ can have zeros; this "singular" RH problem can be mapped to a "regular" RH problem (i.e., to an RH problem for holomorphic functions) coupled with a system of algebraic equations; see [64]. The unique solvability of the relevant algebraic equations and the proof of the associated vanishing lemma are based on the symmetry properties of J; see [64].

Proof that $q(x,t)$ solves the NLS

It is straightforward to prove that if M solves the above RH problem and if $q(x,t)$ is defined by (16.77), then $q(x,t)$ solves the NLS equation. This proof is based on the dressing method; see Chapter 15.

Proof that $q(x,0) = q_0(x)$

Define $M^{(x)}(x,k)$ by

$$M^{(x)} = M(x,0,k), \quad \arg k \in [0, \tfrac{\pi}{2}] \cup [\tfrac{3\pi}{2}, 2\pi]; \tag{16.78a}$$

$$M^{(x)} = M(x,0,k)J_1^{-1}(x,0,k), \quad \arg k \in [\tfrac{\pi}{2}, \pi]; \tag{16.78b}$$

$$M^{(x)} = M(x,0,k)J_3(x,0,k), \quad \arg k \in [\pi, \tfrac{3\pi}{2}]. \tag{16.78c}$$

We first discuss the case that the sets $\{k_j\}$ and $\{\lambda_j\}$ are empty.

The function $M^{(x)}$ is sectionally meromorphic in $\mathbb{C} \setminus \mathbb{R}$. Furthermore,

$$M_-^{(x)}(x,k) = M_+^{(x)}(x,k) J^{(x)}(x,k), \quad k \in \mathbb{R},$$
$$M^{(x)}(x,k) = I + O\left(\frac{1}{k}\right), \quad k \to \infty,$$

where $J^{(x)}(x,t)$ is as defined in (16.45c). Thus according to (16.44),

$$q_0(x) = 2i \lim_{k\to\infty} k\Big(M^{(x)}(x,k)\Big)_{12}.$$

Comparing this equation with (16.77) evaluated at $t = 0$, we conclude that $q_0(x) = q(x,0)$.

We now discuss the case that the sets $\{k_j\}$ and $\{\lambda_j\}$ are *not* empty. The first column of $M(x,t,k)$ has poles at $\{k_j\}_1^{n_1}$ for $\arg k \in (0, \frac{\pi}{2})$ and has poles at $\{\lambda_j\}_1^{\Lambda}$ for $\arg k \in (\frac{\pi}{2}, \pi)$. On the other hand, the first column of $M^{(x)}(x,k)$ should have poles at $\{k_j\}_1^n$, $n = n_1 + n_2$. We will now show that the transformations defined by (16.78) map the former poles to the latter ones. Since $M^{(x)} = M(x,0,k)$ for $\arg k \in [0, \frac{\pi}{2}]$, $M^{(x)}$ has poles at $\{k_j\}_1^{n_1}$ with the correct residue conditions. Letting $M = (M_1, M_2)$, (16.78b) can be written as

$$M^{(x)}(x,k) = \Big(M_1(x,0,k) - \Gamma(k)e^{2ikx}M_2(x,0,k), M_2(x,0,k)\Big).$$

The residue condition at λ_j implies that $M^{(x)}$ has *no* poles at λ_j; on the other hand, this equation shows that $M^{(x)}$ has poles at $\{k_j\}_{n_1+1}^n$ with residues given by

$$\operatorname*{Res}_{k_j} [M^{(x)}(x,k)]_1 = -\operatorname*{Res}_{k_j} \Gamma(k)e^{2ik_jx}[M^{(x)}(x,k_j)]_2, \quad j = n_1+1,\dots,n,$$

which, using the definition of $\Gamma(k)$ and the equation $d(k_j) = -\lambda b(k_j)\overline{B(\bar{k}_j)}$, becomes the residue condition of (16.45e). Similar considerations apply to $\bar{k}_j$ and $\bar{\lambda}_j$.

Proof that $q(0,t) = g_0(t)$ and $q_x(0,t) = g_1(t)$

Let $M^{(1)}(x,t,k),\dots,M^{(4)}(x,t,k)$ denote $M(x,t,k)$ for $\arg k \in [0,\frac{\pi}{2}],\dots,[\frac{3\pi}{2}, 2\pi]$. Recall that M satisfies

$$M^{(2)} = M^{(1)}J_1, \qquad M^{(2)} = M^{(3)}J_2,$$
$$M^{(4)} = M^{(1)}J_4, \qquad M^{(4)} = M^{(3)}J_3, \qquad (J_2 = J_3J_4^{-1}J_1) \tag{16.79}$$

on the respective parts of the contour $\mathcal{L} = \mathbb{R} \cup i\mathbb{R}$; see Figure 16.1.

Let $M^{(t)}(t,k)$ be defined by

$$M^{(t)}(t,k) = M(0,t,k)G(t,k), \tag{16.80}$$

where G is given by $G^{(1)},\dots,G^{(4)}$ for $\arg k \in [0,\frac{\pi}{2}],\dots,[\frac{3\pi}{2},2\pi]$. Suppose we can find matrices $G^{(1)}$ and $G^{(2)}$ holomorphic for $\operatorname{Im} k > 0$ (and continuous for $\operatorname{Im} k \geq 0$), and find matrices $G^{(3)}$ and $G^{(4)}$ holomorphic for $\operatorname{Im} k < 0$ (and continuous for $\operatorname{Im} k \leq 0$), which tend to I as $k \to \infty$, and which satisfy

$$J_1(0,t,k)G^{(2)}(t,k) = G^{(1)}(t,k)J^{(t)}(t,k), \quad k \in i\mathbb{R}^+, \tag{16.81a}$$
$$J_3(0,t,k)G^{(4)}(t,k) = G^{(3)}(t,k)J^{(t)}(t,k), \quad k \in i\mathbb{R}^-, \tag{16.81b}$$
$$J_4(0,t,k)G^{(4)}(t,k) = G^{(1)}(t,k)J^{(t)}(t,k), \quad k \in \mathbb{R}^+, \tag{16.81c}$$

where $J^{(t)}(t,k)$ is the jump matrix in (16.63c). Then equations (16.81) yield

$$J_2(0,t,k)G^{(2)}(t,k) = G^{(3)}(t,k)J^{(t)}(t,k), \quad k \in \mathbb{R}^-,$$

and (16.79) and (16.80) imply that $M^{(t)}$ satisfies the RH problem defined in (16.63). If the sets $\{k_j\}$ and $\{\lambda_j\}$ are empty, this immediately yields the desired result.

The existence of the matrices $G^{(j)}$ is a consequence of the global relation (16.36). Indeed, using this latter equation we will establish the following formulae:

$$G^{(1)} = \begin{pmatrix} \dfrac{a(k)}{A(k)} & c^+(k)e^{4ik^2(T-t)} \\ 0 & \dfrac{A(k)}{a(k)} \end{pmatrix}, \quad G^{(4)} = \begin{pmatrix} \dfrac{\overline{A(\bar{k})}}{\overline{a(\bar{k})}} & 0 \\ \lambda\overline{c^+(\bar{k})}e^{-4ik^2(T-t)} & \dfrac{\overline{a(\bar{k})}}{\overline{A(\bar{k})}} \end{pmatrix},$$

$$G^{(2)} = \begin{pmatrix} d(k) & \dfrac{-b(k)e^{-4ik^2t}}{\overline{A(\bar{k})}} \\ 0 & \dfrac{1}{d(k)} \end{pmatrix}, \qquad G^{(3)} = \begin{pmatrix} \dfrac{1}{\overline{d(\bar{k})}} & 0 \\ \dfrac{-\lambda\overline{b(\bar{k})}}{A(k)}e^{4ik^2t} & \overline{d(\bar{k})} \end{pmatrix}. \quad (16.82)$$

We first verify (16.81a). The (1-2) element is proportional to the global relation; the (2-1) and (2-2) elements are satisfied identically. The (1-1) element is satisfied if and only if

$$d = \frac{a}{A} + \frac{\lambda\bar{B}}{A}c^+e^{4ik^2T}. \quad (16.83)$$

Using $A\bar{A} - \lambda B\bar{B} = 1$, we find

$$d = \frac{a}{A}A\bar{A} - \lambda b\bar{B} = \frac{a}{A}(1 + \lambda B\bar{B}) - \lambda b\bar{B} = \frac{a}{A} + \frac{\lambda\bar{B}}{A}(aB - bA),$$

which equals the RHS of (16.83) in view of the global relation (16.36).

Equation (16.81b) follows from (16.81a) and the symmetry relations

$$G^{(4)}(k) = \sigma_\lambda\overline{G^{(1)}(\bar{k})}\sigma_\lambda, \quad G^{(3)}(k) = \sigma_\lambda\overline{G^{(2)}(\bar{k})}\sigma_\lambda, \quad J_3(k) = \sigma_\lambda\overline{J_1^{-1}(\bar{k})}\sigma_\lambda,$$

where

$$\sigma_\lambda = \begin{cases} \sigma_1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \text{if } \lambda = 1, \\ \sigma_2 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} & \text{if } \lambda = -1. \end{cases}$$

Equation (16.81c) can be verified in a way similar to (16.81a). In fact, in this case one has to use all three basic algebraic identities which hold on the real axis, namely both the determinant relations, $|a|^2 - \lambda|b|^2 = 1$ and $|A|^2 - \lambda|B|^2 = 1$, and the global relation, $a(k)B(k) - b(k)A(k) = c^+(k)e^{4ik^2T}$.

We now consider the case that the sets $\{k_j\}$ and $\{\lambda_j\}$ are *not* empty.

(a) $\arg k \in (0, \frac{\pi}{2})$. Let $M = (M_1, M_2)$; then (16.80) and (16.82) imply

$$M^{(t)}(t,k) = \left(\frac{a(k)}{A(k)} M_1(0,t,k), c^+(k) e^{4ik^2(T-t)} M_1(0,t,k) + \frac{A(k)}{a(k)} M_2(0,t,k)\right).$$

Suppose that $k_0 \in \{k_j\}_1^{n_1}$ and $k_0 \notin \{K_j\}_1^{N_1}$, where $\{K_j\}_1^{N_1}$ denotes the set of zeros of $A(k)$ in the first quadrant. Then, $M^{(t)}(t,k)$ does *not* have a pole at k_0. Indeed,

$$\operatorname*{Res}_{k_0} [M^{(t)}(t,k)]_2 = c^+(k_0) e^{4ik_0^2(T-t)} \operatorname*{Res}_{k_0} M_1(0,t,k) + \frac{A(k_0)}{\dot{a}(k_0)} M_2(0,t,k_0).$$

Using

$$\operatorname*{Res}_{k_0} M_1(0,t,k) = \frac{M_2(0,t,k_0) e^{4ik_0^2 t}}{\dot{a}(k_0) b(k_0)},$$

we find

$$\operatorname*{Res}_{k_0} [M^{(t)}(t,k)]_2 = \frac{M_2(0,t,k_0)}{\dot{a}(k_0) b(k_0)} \left(c^+(k_0) e^{4ik_0^2 T} + b(k_0) A(k_0)\right).$$

From the global relation, the term in the parentheses equals $a(k_0)B(k_0)$, and hence

$$\operatorname*{Res}_{k_0} [M^{(t)}(t,k)]_2 = 0.$$

Suppose that $K_0 \in \{K_j\}_1^{N_1}$ and $K_0 \notin \{k_j\}_1^{n_1}$. Then, $[M^{(t)}(t,k)]_1$ has a pole at K_0. In order to compute the associated residues we note that

$$\operatorname*{Res}_{K_0} [M^{(t)}(t,k)]_1 = \frac{a(K_0)}{\dot{A}(K_0)} M_1(0,t,K_0).$$

Using the definition of the second column of $M^{(t)}$ evaluated at $k = K_0$,

$$M_1(0,t,K_0) = \frac{e^{4iK_0^2 t} [M^{(t)}(t,K_0)]_2}{c^+(K_0) e^{4iK_0^2 T}},$$

and the global relation evaluated at $k = K_0$,

$$a(K_0) B(K_0) = c^+(K_0) e^{4iK_0^2 T},$$

we find

$$\operatorname*{Res}_{K_0} [M^{(t)}(t,k)]_1 = \frac{e^{4iK_0^2 t} [M^{(t)}(t,K_0)]_2}{\dot{A}(K_0) B(K_0)},$$

which is the residue condition in (16.63e). Note that since K_0 is not a common zero for $a(k)$ and $A(k)$, it follows that $c^+(K_0) \neq 0$.

Suppose now that $k_0 \equiv K_0$ is a common (simple) zero of the functions $a(k)$ and $A(k)$. Then necessarily

$$c^+(k_0) = 0, \tag{16.84}$$

and the second column of $M^{(t)}(t,k)$ does not have a pole at k_0. The first column has a pole at $k_0 \equiv K_0$, and for the residue condition we have

$$\operatorname*{Res}_{K_0}[M^{(t)}(t,k)]_1 = \frac{\dot a(K_0)}{\dot A(K_0)}\operatorname*{Res}_{K_0} M_1(0,t,k) = \frac{e^{4iK_0^2 t}}{\dot A(K_0) b(K_0)} M_2(0,t,K_0). \tag{16.85}$$

Using, as before, the definition of the second column of $M^{(t)}$ evaluated at $k = K_0$, we obtain the equation

$$\begin{aligned}[M^{(t)}(t,k)]_2 &= \dot c^+(K_0)e^{4iK_0^2(T-t)}\operatorname*{Res}_{K_0} M_1(0,t,k) + \frac{\dot A(K_0)}{\dot a(K_0)}M_2(0,t,K_0)\\ &= M_2(0,t,K_0)\left(\frac{\dot A(K_0)}{\dot a(K_0)} + \frac{\dot c^+(K_0)e^{4iK_0^2T}}{\dot a(K_0)b(K_0)}\right) = M_2(0,t,K_0)\frac{B(K_0)}{b(K_0)},\end{aligned} \tag{16.86}$$

where in the last step we have used the equation

$$\dot c^+(K_0)e^{4iK_0^2T} = \dot a(K_0)B(K_0) - \dot A(K_0)b(K_0),$$

which follows from the global relation and from (16.84). By virtue of (16.86), equation (16.85) can be rewritten as

$$\operatorname*{Res}_{K_0}[M^{(t)}(t,k)]_1 = \frac{e^{4iK_0^2t}}{\dot A(K_0)B(K_0)}[M^{(t)}(t,K_0)]_2,$$

which again reproduces the residue condition in (16.63e).

We note that the last arguments, further simplified by $\dot c^+(K_0)e^{4iK_0^2T} \mapsto 0$, are precisely the ones we need in the case $T=\infty$, when the global relation takes the form (16.75) so that $\{k_j\}_1^{n_1} = \{K_j\}_1^{N_1}$.

(b) $\arg k \in (\frac{\pi}{2},\pi)$ Equations (16.80) and (16.82) imply

$$M^{(t)}(t,k) = \left(d(k)M_1(0,t,k),\ -\frac{b(k)}{\overline{A(\bar k)}}e^{-4ik^2t}M_1(0,t,k) + \frac{M_2(0,t,k)}{d(k)}\right).$$

Suppose that $\lambda_0 \in \{\lambda_j\}_1^\Lambda$ and $\lambda_0 \notin \{\bar K_j\}_{N_1+1}^N$, where $\{K_j\}_{N_1+1}^N$ denotes the set of zeros of $A(k)$ in the third quadrant. Then, $M^{(t)}(t,k)$ does *not* have a pole at λ_0. Indeed,

$$\operatorname*{Res}_{\lambda_0}[M^{(t)}(t,k)]_2 = \frac{-b(\lambda_0)}{\overline{A(\bar\lambda_0)}}e^{-4i\lambda_0^2t}\operatorname*{Res}_{\lambda_0} M_1(0,t,k) + \frac{M_2(0,t,\lambda_0)}{\dot d(\lambda_0)}.$$

Using

$$\operatorname*{Res}_{\lambda_0} M_1(0,t,k) = \frac{\lambda\overline{B(\bar\lambda_0)}e^{4i\lambda_0^2t}M_2(0,t,\lambda_0)}{a(\lambda_0)\dot d(\lambda_0)} \tag{16.87}$$

and taking into account that under the assumption on λ_0,

$$d(\lambda_0) = 0 \quad\Longrightarrow\quad \frac{\lambda\overline{B(\bar\lambda_0)}}{a(\lambda_0)} = \frac{\overline{A(\bar\lambda_0)}}{b(\lambda_0)},$$

it follows that $\operatorname*{Res}_{\lambda_j} [M^{(t)}(t,k)]_2 = 0$.

Suppose that $K_0 \in \{K_j\}_{N_1+1}^{N}$ and $\bar{K}_0 \notin \{\lambda_j\}_1^{\Lambda}$. Then, $[M^{(t)}(t,k)]_2$ has a pole at $\bar{K}_0$. In order to compute the associated residues we note that

$$\operatorname*{Res}_{\bar{K}_0} [M^{(t)}(t,k)]_2 = \frac{-b(\bar{K}_0)}{\overline{\dot{A}}(K_0)} e^{-4i\bar{K}_0^2 t} M_1(0,t,\bar{K}_0).$$

Using the definition of the first column of $M^{(t)}$ at $k = \bar{K}_0$ and recalling that $d(\bar{K}_0) = -\lambda \overline{B}(K_0) b(\bar{K}_0)$ (and hence, in particular, $\overline{B}(K_0) b(\bar{K}_0) \neq 0$), we find

$$[M^{(t)}(t,\bar{K}_0)]_1 = -\lambda \overline{B}(K_0) b(\bar{K}_0) M_1(0,t,\bar{K}_0).$$

Thus

$$\operatorname*{Res}_{\bar{K}_0} [M^{(t)}(t,k)]_2 = \frac{\lambda e^{-4i\bar{K}_0^2 t}[M^{(t)}(t,\bar{K}_0)]_1}{\overline{\dot{A}}(K_0)\overline{B}(K_0)},$$

which is the residue condition in (16.63e).

Suppose now that $\lambda_0 \equiv \bar{K}_0$ is a common (simple) zero of the functions $d(k)$ and $\overline{A(\bar{k})}$. Then necessarily

$$b(\lambda_0) = 0,$$

and for the residue of $[M^{(t)}(t,k)]_2$ at $\bar{K}_0$ we have

$$\begin{aligned} \operatorname*{Res}_{\bar{K}_0} [M^{(t)}(t,k)]_2 &= \frac{-\dot{b}(\bar{K}_0)}{\overline{\dot{A}}(K_0)} e^{-4i\bar{K}_0^2 t} \operatorname*{Res}_{\bar{K}_0} M_1(0,t,k) + \frac{M_2(0,t,\bar{K}_0)}{\dot{d}(\bar{K}_0)} \\ &= \frac{1}{\overline{\dot{A}}(K_0) a(\bar{K}_0)} M_2(0,t,\bar{K}_0), \end{aligned} \tag{16.88}$$

where we have used the residue condition (16.87) for $M_1(0,t,k)$ at $\lambda_0 \equiv \bar{K}_0$ and the equation

$$\dot{d}(\bar{K}_0) = \overline{\dot{A}}(K_0) a(\bar{K}_0) - \lambda \overline{B}(K_0) \dot{b}(\bar{K}_0).$$

Using the definition of the first column of $M^{(t)}$ at $k = \bar{K}_0$ and the residue equation (16.87) once more, we conclude that

$$M_2(0,t,\bar{K}_0) = \lambda \frac{a(\bar{K}_0)}{\overline{B}(K_0)} e^{-4i\bar{K}_0^2 t}[M^{(t)}(t,\bar{K}_0)]_1,$$

which, together with (16.88), yields again the residue condition in (16.63e).

Similar considerations are valid for $\arg k \in [\frac{3\pi}{2}, 2\pi]$ and $\arg k \in [\pi, \frac{3\pi}{2}]$. Alternatively, one can use the symmetry relations generated by the transformation $k \mapsto \bar{k}$. ∎

Remark 16.7. In the case $T = \infty$, the matrices $G^{(j)}(t,k)$ are defined and analytic only in the respective quadrants of the complex plane k. Moreover, the global relation holds only in the first quadrant (see (16.75)); in this case the relation $J_4(0,t,k)G^{(2)}(t,k) = G^{(3)}(t,k)J^{(t)}(t,k)$, $k < 0$, can be verified independently, with the use of the determinant relations.

Remark 16.8. It is shown in [50] that the solution $q(x,t)$ for $0 < t < T_*$, where $0 < T_* < T$, depends only on the boundary values for t between 0 and T_*.

16.2 The Modified KdV, KdV, and sG Equations

In this section we will study the following equations.

(a) The modified KdV (mKdV) equation:

$$\frac{\partial q}{\partial t} - \frac{\partial^3 q}{\partial x^3} + 6\lambda q^2 \frac{\partial q}{\partial x} = 0, \quad \lambda = \pm 1, \quad q \text{ real.} \tag{16.89}$$

Usually the mKdV equation occurs with the plus sign in front of q_{xxx}, thus we will refer to (16.89) as the mKdVII equation.

(b) The KdV equation:

$$\frac{\partial q}{\partial t} + \frac{\partial q}{\partial x} - \frac{\partial^3 q}{\partial x^3} + 6q\frac{\partial q}{\partial x} = 0, \quad q \text{ real.} \tag{16.90}$$

Similarly, we will refer to this equatoin as the KdVII equation.

(c) The sG equation in laboratory conditions:

$$\frac{\partial^2 q}{\partial t^2} - \frac{\partial^2 q}{\partial x^2} + \sin q = 0, \quad q \text{ real.} \tag{16.91}$$

The above equations admit the following Lax pair formulation:

$$\mu_x + i f_1(k)\hat{\sigma}_3\mu = Q(x,t,k)\mu, \tag{16.92a}$$

$$\mu_t + i f_2(k)\hat{\sigma}_3\mu = \tilde{Q}(x,t,k)\mu, \quad k \in \mathbb{C}, \tag{16.92b}$$

where the functions f_1, f_2, Q, $\tilde{Q}$ are given by the following expressions:

(a) $f_1 = -k,\ f_2 = 4k^3$,

$$Q = \begin{pmatrix} 0 & q(x,t) \\ \lambda q(x,t) & 0 \end{pmatrix},$$

$$\tilde{Q} = \begin{pmatrix} -2i\lambda k q^2 & -4k^2 q + 2ikq_x - 2\lambda q^3 + q_{xx} \\ \lambda(-4k^2 q - 2ikq_x - 2\lambda q^3 + q_{xx}) & 2i\lambda k q^2 \end{pmatrix}. \tag{16.93}$$

(b) $f_1 = -k, \; f_2 = k + 4k^3$,

$$Q(x,t,k) = \frac{q}{2k}(\sigma_2 - i\sigma_3),$$

$$\tilde{Q}(x,t,k) = -2kq\sigma_2 + q_x\sigma_1 + \frac{2q^2 + q - q_{xx}}{2k}(i\sigma_3 - \sigma_2). \tag{16.94}$$

(c) $f_1 = \frac{1}{4}(k - \frac{1}{k}), \; f_2 = \frac{1}{4}(k + \frac{1}{k})$,

$$Q(x,t,k) = -\frac{i}{4}(q_x + q_t)\sigma_1 - \frac{1}{4k}(\sin q)\sigma_2 + \frac{i}{4k}\left[(\cos q) - 1\right]\sigma_3,$$

$$\tilde{Q}(x,t,k) = Q(x,t,-k). \tag{16.95}$$

In (16.94) and (16.95), $\{\sigma_j\}_1^3$ denote the usual Pauli matrices, i.e.,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{16.96}$$

In what follows we will implement the first two steps needed for the analysis of the nonlinear integrable PDEs (16.89)–(16.91). The derivation of the relevant results are very similar to those presented in section 16.1, and thus the details will be omitted (details can be found in [51], [53]). Also, for brevity of presentation we will *not* present the residue conditions; i.e., we consider only the solitoness case. Solitons for the mKdV are discussed in [53]. Using the formulae of [50] it is straightforward to add the solitonic part to the formulae for the KdVII and the sG: The zeros for $0 < \arg k < \pi/2$ and $\pi/2 < \arg k < \pi$ in the NLS can occur in the domains D_1 and D_2 for the sG and the KdVII.

16.2.1. Assume That *q*(*x*,*t*) Exists

Equations (16.92) can be rewritten in the form

$$d\left[e^{i(f_1(k)x + f_2(k)t)\hat{\sigma}_3}\mu(x,t,k)\right] = W(x,t,k), \quad k \in \mathbb{C}, \tag{16.97}$$

where $W(x,t,k)$ is defined by

$$W(x,t,k) = e^{i(f_1(k)x + f_2(k)t)\hat{\sigma}_3}\left(Q(x,t,k)\mu(x,t,k)dx + \tilde{Q}(x,t,k)\mu(x,t,k)dt\right). \tag{16.98}$$

16.2.1.1 The Direct Problem

We define $\{\mu_j(x,t,k)\}_1^3$ by

$$\mu_j(x,t,k) = I + \int_{(x_j,t_j)}^{(x,t)} e^{-i(f_1(k)\xi + f_2(k)\tau)\hat{\sigma}_3} W(\xi,\tau,k), \quad (x,t) \in \Omega, \tag{16.99}$$

where $(x_j, t_j)_1^3$ are chosen as in section 10.1; see Figure 10.1.

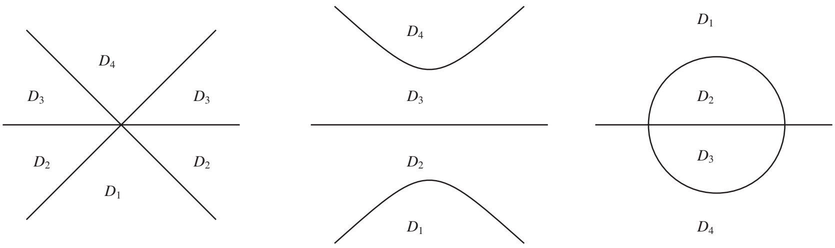


Figure 16.3. *The domains D_j, $j = 1, \dots, 4$ for the mKdVII, KdVII, and sG equations.*

The collection of the matrices $\{\mu_j\}_1^3$; (see (16.7)) provides the solution of the direct problem, where the superscripts (1), …, (4), instead of denoting the first, …, fourth quadrant of the complex k-plane, now denote the domains $\{D_j\}_1^4$ which are defined as follows:

$$D_1 = \{k \in \mathbb{C}, \operatorname{Im} f_1(k) > 0 \cap \operatorname{Im} f_2(k) > 0\},$$

$$D_2 = \{k \in \mathbb{C}, \operatorname{Im} f_1(k) > 0 \cap \operatorname{Im} f_2(k) < 0\},$$

$$D_3 = \{k \in \mathbb{C}, \operatorname{Im} f_1(k) < 0 \cap \operatorname{Im} f_2(k) > 0\},$$

$$D_4 = \{k \in \mathbb{C}, \operatorname{Im} f_1(k) < 0 \cap \operatorname{Im} f_2(k) < 0\}. \tag{16.100}$$

The definitions of f_1, f_2 and the definitions of the domains $\{D_j\}_1^4$ imply that for (16.89)–(16.91) these domains are given as follows (see Figure 16.3):

(a)

$$D_1 = \left\{\frac{4\pi}{3} < \arg k < \frac{5\pi}{3}\right\}, \quad D_2 = \left\{\left(\pi < \arg k < \frac{4\pi}{3}\right) \cup \left(\frac{5\pi}{3} < \arg k < 2\pi\right)\right\},$$

$$D_3 = \left\{\left(0 < \arg k < \frac{\pi}{3}\right) \cup \left(\frac{2\pi}{3} < \arg k < \pi\right)\right\}, \quad D_4 = \left\{\frac{\pi}{3} < \arg k < \frac{2\pi}{3}\right\}. \tag{16.101}$$

(b) Let the curves $l_\pm$ be defined by

$$l_\pm = \left\{k = k_R + ik_I, k_I \gtrless 0, \quad \frac{1}{4} + 3k_R^2 - k_I^2 = 0\right\},$$

$$D_1 = \{\operatorname{Im} k < \operatorname{Im} k_-\}, \quad D_2 = \{\operatorname{Im} k_- < \operatorname{Im} k < 0\}, \quad k_- \in l_-,$$

$$D_3 = \{0 < \operatorname{Im} k < \operatorname{Im} k_+\}, \quad D_4 = \{\operatorname{Im} k > \operatorname{Im} k_+\}, \quad k_+ \in l_+. \tag{16.102}$$

(c)

$$D_1 = \{\operatorname{Im} k > 0 \cap |k| > 1\}, \quad D_2 = \{\operatorname{Im} k > 0 \cap |k| < 1\},$$

$$D_3 = \{\operatorname{Im} k < 0 \cap |k| < 1\}, \quad D_4 = \{\operatorname{Im} k < 0 \cap |k| > 1\}. \tag{16.103}$$

16.2.1.2 The Inverse Problem

The "jump matrix" of the relevant RH problem needed for the formulation of the inverse problem is uniquely defined in terms of the following 2×2 matrix-valued functions:

$$s(k) = \mu_3(0,0,k), \quad S(k) = \left(e^{if_2(k)T\hat{\sigma}_3}\mu_2(0,T,k)\right)^{-1}. \tag{16.104}$$

This is a direct consequence of the fact that any two solutions of (16.99) are simply related. In particular,

$$\mu_3(x,t,k) = \mu_2(x,t,k)e^{-i(f_1(k)x+f_2(k)t)\hat{\sigma}_3}\mu_3(0,0,k),$$
$$\mu_1(x,t,k) = \mu_2(x,t,k)e^{-i(f_1(k)x+f_2(k)t)\hat{\sigma}_3}\left(e^{if_2(k)T\hat{\sigma}_3}\mu_2(0,T,k)\right)^{-1}. \tag{16.105}$$

The functions $s(k)$ and $S(k)$ follow from the evaluation at $x = 0$ and at $t = T$ of the function $\mu_3(x,0,k)$ and $\mu_2(0,t,k)$, respectively. These latter functions satisfy the following linear integral equations:

$$e^{if_1(k)x\hat{\sigma}_3}\mu_3(x,0,k) = I - \int_x^\infty e^{if_1(k)\xi\hat{\sigma}_3}(Q\mu_3)(\xi,0,k)d\xi, \tag{16.106}$$

$$e^{if_2(k)t\hat{\sigma}_3}\mu_2(0,t,k) = I + \int_0^t e^{if_2(k)\tau\hat{\sigma}_3}(\tilde{Q}\mu_2)(0,\tau,k)d\tau. \tag{16.107}$$

The functions $\mu_3(x,0,k)$ and $\mu_2(0,t,k)$, and hence the functions $s(k)$ and $S(k)$, are uniquely defined in terms of $Q(x,0,k)$ and $\tilde{Q}(0,t,k)$, i.e., in terms of the initial conditions and of the boundary values, respectively.

The Spectral Functions

For the mKdVII, the KdVII, and the sG with $q(x,t)$ real, the matrices Q and $\tilde{Q}$ have certain symmetry properties. These symmetries imply the following symmetries for μ:

$$(\mu(x,t,k))_{11} = \overline{(\mu(x,t,\bar{k}))_{22}}, \quad (\mu(x,t,k))_{12} = \rho\overline{(\mu(x,t,\bar{k}))_{21}}, \tag{16.108a}$$

where $\rho = \lambda$ for the mKdVII, $\rho = 1$ for the KdVII, and $\rho = -1$ for the sG.

In addition, the following symmetries are valid:

$$(\mu(x,t,k))_{11} = (\mu(x,t,-k))_{22}, \quad (\mu(x,t,k))_{12} = (\mu(x,t,-k))_{21}. \tag{16.108b}$$

The fact that Q and $\tilde{Q}$ are traceless, together with (16.8), implies

$$\det \mu(x,t,k) = 1. \tag{16.109}$$

The functions $\mu_3(x,0,k)$ and $\mu_2(0,t,k)$ have larger domains of analyticity as follows:

$$\mu_1(0,t,k) = \left(\mu_1^{(24)}(0,t,k), \quad \mu_1^{(13)}(0,t,k)\right),$$

$$\mu_2(0,t,k) = \left(\mu_2^{(13)}(0,t,k), \ \mu_2^{(24)}(0,t,k)\right).$$

The symmetry properties (16.108a) imply similar symmetry properties for the spectral functions. This justifies the following notations for the spectral functions:

$$s(k) = \begin{pmatrix} \overline{a(\bar{k})} & b(k) \\ \rho\overline{b(\bar{k})} & a(k) \end{pmatrix}, \quad S(k) = \begin{pmatrix} \overline{A(\bar{k})} & B(k) \\ \rho\overline{B(\bar{k})} & A(k) \end{pmatrix}, \tag{16.110}$$

where

$$\rho = \lambda \text{ for the mKdVII}, \quad \rho = 1 \text{ for the KdVII}, \quad \rho = -1 \text{ for the sG}.$$

These notations and the definitions of $s(k)$ and $S(k)$, i.e., equations (16.104), imply

$$\begin{pmatrix} b(k) \\ a(k) \end{pmatrix} = \mu_3^{(12)}(0,0,k), \qquad \begin{pmatrix} -e^{-2f_2(k)T}B(k) \\ \overline{A(\bar{k})} \end{pmatrix} = \mu_2^{(24)}(0,T,k),$$

where the vectors $\mu_3^{(12)}(x,0,k)$, $\mu_2^{(24)}(0,t,k)$ satisfy the following ODEs:

$$\partial_x\mu_3^{(12)}(x,0,k) + 2if_1(k)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\mu_3^{(12)}(x,0,k) = Q(x,0,k)\mu_3(x,0,k),$$
$$k \in \bar{D}_1 \cup \bar{D}_2, 0 < x < \infty,$$
$$\lim_{x\to\infty}\mu_3^{(12)}(x,0,k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and

$$\partial_t\mu_2^{(24)}(x,0,k) + 2if_2(k)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\mu_2^{(24)}(0,t,k) = \tilde{Q}(0,t,k)\mu_2(0,t,k),$$
$$k \in \bar{D}_2 \cup \bar{D}_4, 0 < t < T,$$
$$\mu_2^{(24)}(0,0,k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

where $\bar{D}$ denotes the union of D and its boundary.

The above definitions imply the following properties:

$a(k)$, and $b(k)$

- $a(k)$, $b(k)$ are defined and are analytic for $k \in D_1 \cup D_2$.
- $$|a(k)|^2 - \rho|b(k)|^2 = 1, k \in \mathbb{R}. \tag{16.111a}$$
- $$a(k) = 1 + O\left(\frac{1}{k}\right), b(k) = O\left(\frac{1}{k}\right), k \to \infty. \tag{16.111b}$$

$A(k)$, and $B(k)$

- $A(k)$, $B(k)$ are entire functions which are bounded for $k \in D_1 \cup D_3$; if $T = \infty$ these functions are defined and are analytic for k in this domain.
- $A(k)\overline{A(\bar{k})} - \rho B(k)\overline{B(\bar{k})} = 1, k \in \mathbb{C}$.
- $$A(k) = 1 + O\left(\frac{1 + e^{2if_2(k)T}}{k}\right), \quad B(k) = O\left(\frac{e^{2if_2(k)T}}{k}\right), \quad k \to \infty. \quad (16.112)$$

For the sG and the KdVII equations the above are valid in the punctured complex k-plane, $k \in \mathbb{C}\backslash\{0\}$.

All of the above properties, except for the property that $B(k)$ is bounded for $k \in D_1 \cup D_3$, follow from the analyticity properties of $\mu_3(x, 0, k)$, $\mu_2(0, t, k)$, from the conditions of unit determinant, and from the large k asymptotics of these eigenfunctions. Regarding $B(k)$ we note that $B(k) = B(T, k)$, where $B(t, k) = -\exp[2if_2(k)t](\mu_2^{24}(0, t, k))_1$. The vector $\exp(2if_2(k)t)\mu_2^{(24)}(0, t, k)$ satisfies a linear Volterra integral equation, from which it immediately follows that $B(t, k)$ is an entire function of k bounded for $k \in D_1 \cup D_3$.

The RH Problem

Equations (16.105) can be rewritten in the form

$$M(x, t, k) = M_+(x, t, k)J(x, t, k), \quad k \in \mathcal{L}, \quad (16.113)$$

where the matrices M_-, M_+, J and the oriented contour $\mathcal{L}$ are defined as follows:

$$M_+ = \left(\frac{\mu_2^{(1)}}{a(k)}, \mu_3^{(12)}\right), \quad k \in D_1; \quad M_- = \left(\frac{\mu_1^{(2)}}{d(k)}, \mu_3^{(12)}\right), \quad k \in D_2;$$

$$M_+ = \left(\mu_3^{(34)} \frac{\mu_1^{(3)}}{\overline{\overline{d(\bar{k})}}}\right), \quad k \in D_3; \quad M_- = \left(\mu_3^{(34)} \frac{\mu_2^{(4)}}{\overline{\overline{a(\bar{k})}}}\right), \quad k \in D_4; \quad (16.114)$$

$$d(k) = a(k)\overline{A(\bar{k})} - \rho b(k)\overline{B(\bar{k})}. \quad (16.115)$$

$$J(x, t, k) = \begin{cases} J_1, & k \in D_1 \cap D_2 \doteqdot \mathcal{L}_1, \\ J_2, & k \in D_2 \cap D_3 \doteqdot \mathcal{L}_2, \\ J_3, & k \in D_3 \cap D_4 \doteqdot \mathcal{L}_3, \\ J_4, & k \in D_4 \cap D_1 \doteqdot \mathcal{L}_4, \quad J_2 = J_3 J_4^{-1} J_1. \end{cases} \quad (16.116)$$

$$J_1 = \begin{pmatrix} 1 & 0 \\ \Gamma(k)e^{2i\theta} & 1 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 1 & -\rho\overline{\Gamma(\bar{k})}e^{-2i\theta} \\ 0 & 1 \end{pmatrix},$$

$$J_4 = \begin{pmatrix} 1 & -\gamma(k)e^{-2i\theta} \\ \rho\bar{\gamma}(k)e^{2i\theta} & 1-\rho|\gamma(k)|^2 \end{pmatrix}, \tag{16.117}$$

where

$$\gamma(k) = \frac{b(k)}{\overline{a(\bar{k})}}; \quad \theta(x,t,k) = f_1(k)x + f_2(k)t, \tag{16.118}$$

$$\Gamma(k) = \frac{\rho \overline{B(\bar{k})}/\overline{A(\bar{k})}}{a(k)[a(k) - \rho b(k)\overline{B(\bar{k})}/\overline{A(\bar{k})}]}, \quad k \in D_2.$$

In order to derive (16.113) we write (16.105) in the form

$$\left(\mu_3^{(34)}, \mu_3^{(12)}\right) = \left(\mu_2^{(1)}, \mu_2^{(4)}\right) \begin{pmatrix} \bar{a} & be^{-2i\theta} \\ \rho\bar{b}e^{2i\theta} & a \end{pmatrix}, \tag{16.119}$$

$$\left(\mu_1^{(2)}, \mu_1^{(3)}\right) = \left(\mu_2^{(1)}, \mu_2^{(4)}\right) \begin{pmatrix} \bar{A} & Be^{-2i\theta} \\ \rho\bar{B}e^{2i\theta} & A \end{pmatrix}. \tag{16.120}$$

In order to compute J_4 we must relate those eigenfunctions which are bounded in D_1 and in D_4; thus rearranging (16.119) and using (16.111a) we find (16.113) with $J = J_4$ and M_-, M_+ given by the fourth and first of equations (16.114) respectively. Similarly, in order to compute J_1 we must relate those eigenfunctions which are bounded in D_1 and in D_2; thus eliminating $\mu_2^{(4)}$ from the second column of (16.119) and from the first column of (16.120) we find (16.113) with $J = J_1$ and M_-, M_+ given by the second and the first of equations (16.114) respectively. The computation of J_3 follows from the elimination of $\mu_2^{(1)}$ from the first column of (16.119) and from the second column of (16.120).

The jump condition (16.113), together with the analyticity properties and the large k behavior of μ_j, defines a 2×2 matrix RH problem for the determination of the matrix $M(x,t,k)$. This is in general a meromorphic function of k in $\mathbb{C}\backslash\mathcal{L}$. The possible poles of M are generated by the zeros of $a(k)$, $k \in D_1$, and of $d(k)$, $k \in D_2$, and from the complex conjugates of these zeros. For compactness of presentation we assume that no such zeros occur.

By substituting the large k asymptotics expansion of M, i.e., (16.33a), in the x-part of the Lax pair, it is straightforward to obtain an expression for q, i.e., the analogue of (16.34). This expression will be given in Theorem 16.2.

16.2.1.3 The Global Relation

In analogy with (16.35) we now have

$$-I + S(k)^{-1}s(k) + e^{if_2(k)T\hat{\sigma}_3}\int_0^\infty e^{if_1(k)\xi\hat{\sigma}_3}(Q\mu_3)(\xi,T,k)d\xi = 0, k \in (\bar{D}_3 \cup \bar{D}_4, \bar{D}_1 \cup \bar{D}_2), \tag{16.121}$$

where the notation $k \in (D_1, D_2)$ means that the first vector is valid for $k \in D_1$ and the second vector is valid for $k \in D_2$.

The (1-2) element of (16.121) is

$$a(k)B(k) - b(k)A(k) = e^{2ik^2T}c(k), \quad k \in \bar{D}_1 \cup \bar{D}_2, \tag{16.122}$$

where the scalar function

$$c(k) = \int_0^\infty e^{2if_1(k)\xi}(Q\mu_3)_{12}(\xi, T, k)d\xi, \quad k \in \bar{D}_1 \cup \bar{D}_2, \tag{16.123}$$

is of $O(1/k)$ as $k \to \infty$.

16.2.2. Assume That the Spectral Functions Satisfy the Global Relation

In this section we implement step 2 of the three steps introduced at the beginning of Chapter 16; in this respect we first define the spectral functions.

Definition 16.3 (the spectral functions $a(k), b(k)$). For the mKdVII and the KdVII let $q_0(x) \in S(\mathbb{R}^+)$, and for the sG let $q_0(x) - 2\pi m \in S(\mathbb{R}^+)$ and $q_1(x) \in S(\mathbb{R}^+)$, where m is an integer. Let the domains D_j, $j = 1, \dots, 4$, be defined in (16.101)–(16.103). The map

$$\mathbb{S} : \begin{array}{c}\{q_0(x)\} \\ \text{or} \\ \{q_0(x), q_1(x)\}\end{array} \Longrightarrow \{a(k), b(k)\} \tag{16.124}$$

is defined as follows:

$$\begin{pmatrix} b(k) \\ a(k) \end{pmatrix} = \varphi(0, k), \tag{16.125}$$

where the vector-valued function $\varphi(x, k)$ is defined in terms of $q_0(x)$ or $\{q_0(x), q_1(x)\}$ by

$$\partial_x \varphi(x, k) + 2if_1(k)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\varphi(x, k) = Q(x, k)\varphi(x, k), 0 < x < \infty, k \in \bar{D}_1 \cup \bar{D}_2,$$

$$\lim_{x \to \infty} \varphi(x, k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{16.126}$$

and $Q(x,k)$ is given for the mKdVII, KdVII, sG, respectively, by

$$Q(x) = \begin{pmatrix} 0 & q_0(x) \\ \lambda q_0(x) & 0 \end{pmatrix},$$
$$Q(x,k) = \frac{q_0(x)}{2k}(\sigma_2 - i\sigma_3),$$
$$Q(x,k) = -\frac{i}{4}\left(\frac{dq_0(x)}{dx} + q_1(x)\right)\sigma_1 - \frac{i}{4k}\Big(\sin q_0(x)\Big)\sigma_2 + \frac{i}{4k}\Big(\cos q_0(x) - 1\Big)\sigma_3. \tag{16.127}$$

Proposition 16.3 (properties of $a(k)$, $b(k)$). The functions $a(k)$ and $b(k)$ defined above have the following properties:

1. $a(k), b(k)$ are analytic and bounded for $k \in D_1 \cup D_2$.
2. $|a(k)|^2 - \rho|b(k)|^2 = 1, k \in \mathbb{R}$.
3. $a(k) = 1 + O(\frac{1}{k}), b(k) = O(\frac{1}{k}), k \to \infty$.
4. The inverse of the map (16.124) denoted by $\mathbb{Q}$ can be defined for the mKdVII, KdVII, and sG, respectively, as follows:

$$q_0(x) = -2i \lim_{k\to\infty}\left(kM^{(x)}(x,k)\right)_{12},$$

$$q_0(x) = -2i \lim_{k\to\infty}\partial_x\left(kM^{(x)}(x,k)\right)_{22},$$

$$\begin{cases} \cos q_0(x) = 1 + 2\lim\limits_{k\to\infty}\left[\left(kM^{(x)}(x,k)\right)^2_{12} + 2i\partial_x\left(kM^{(x)}(x,k)\right)_{22}\right], \\ q_1(x) = -\dfrac{d}{dx}q_0(x) - 2\lim\limits_{k\to\infty}\left(kM^{(x)}(x,k)\right)_{12}, \end{cases} \tag{16.128}$$

where $M^{(x)}(x,k)$ is the unique solution of the following RH problem:

- $$M^{(x)}(x,k) = \begin{cases} M_+^{(x)}(x,k), & k \in D_1 \cup D_2, \\ M_-^{(x)}(x,k), & k \in D_3 \cup D_4, \end{cases}$$

  is a meromorphic function of k for $k \in \mathbb{C}\backslash\mathbb{R}$.

- $$M^{(x)}(x,k) = I + O\left(\frac{1}{k}\right), \quad k \to \infty.$$

- $$M_-^{(x)}(x,k) = M_+^{(x)}(x,k) J^{(x)}(x,k), \quad k \in \mathbb{R},$$

  where

$$J^{(x)}(x,k) = \begin{pmatrix} 1 & -\frac{b(k)}{\bar{a}(k)} e^{-2if_1(k)} \\ \rho \frac{\bar{b}(k)}{a(k)} e^{2if_1(k)} & \frac{1}{|a|^2} \end{pmatrix}, \quad k \in \mathbb{R}. \tag{16.129}$$

- M satisfies appropriate residue conditions if $a(k)$ has zeros for $k \in D_1 \cup D_2$.

5. $\mathbb{S}^{-1} = \mathbb{Q}$.

6. For the KdVII,

$$a(k) = \frac{i\alpha}{k} + O(1), \quad b(k) = -\frac{i\alpha}{k} + O(1), \quad k \to 0, \tag{16.130}$$

where α is a real constant.

Proof. The derivation of these results, except property 6, is similar to the derivation of the analogous results of Proposition 16.1. The derivation of property 6 is presented in the appendix of this chapter. □

Definition 16.4 (the spectral functions $(A(k), B(k))$). Let $\{g_l(t)\}_0^{n-1}$ be smooth functions for $0 < t < T$, where $n = 2$ for the sG and $n = 3$ for the mKdVII and KdVII equations. Let the domains D_j, $j = 1, \dots, 4$, be defined in (16.101)–(16.103). The map

$$\tilde{\mathbb{S}} : \{g_l(t)\}_0^{n-1} \to \{A(k), B(k)\} \tag{16.131}$$

is defined as

$$\begin{pmatrix} -e^{-2if_2(k)T} B(k) \\ \overline{A(\bar{k})} \end{pmatrix} = \Phi(T,k), \tag{16.132}$$

where the vector-valued function $\Phi(t,k)$ is defined in terms of $\{g_l(t)\}_0^{n-1}$ by

$$\partial_t \Phi(t,k) + 2if_2(k) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Phi(t,k) = \tilde{Q}(t,k)\Phi(t,k), \quad 0 < t < T, \quad k \in D_2 \cup D_4,$$

$$\Phi(0,k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{16.133}$$

and $\tilde{Q}(t,k)$ is given for the mKdVII, KdVII, sG, respectively, by the following formulae:

$$\tilde{Q}(t,k) = \begin{pmatrix} 0 & -4k^2 g_0(t) - 2\lambda(g_0(t))^3 \\ -4k^2\lambda g_0(t) - 2(g_0(t))^3 & 0 \end{pmatrix}$$

$$+2ik\begin{pmatrix} 0 & g_1(t) \\ -g_1(t) & 0 \end{pmatrix}\sigma_3 + \begin{pmatrix} 0 & g_2(t) \\ g_2(t) & 0 \end{pmatrix} - 2i\lambda k(g_0(t))^2\sigma_3,$$

$$\tilde{Q}(t,k) = -2kg_0(t)\sigma_2 + g_1(t)\sigma_1 + \frac{1}{2k}[2g_0(t)^2 + g_0(t) - g_2(t)](i\sigma_3 - \sigma_2),$$

$$\tilde{Q}(t,k) = -\frac{i}{4}\left(\frac{dg_0(t)}{dt} + g_1(t)\right)\sigma_1 + \frac{i}{4k}[\sin g_0(t)]\sigma_2 - \frac{i}{4k}[\cos g_0(t) - 1]\sigma_3. \quad (16.134)$$

Proposition 16.4 (properties of $A(k)$, $B(k)$). The functions $A(k)$ and $B(k)$ defined above have the following properties:

1. $A(k)$, $B(k)$ are entire functions which are bounded for $k \in D_2 \cup D_4$. If $T = \infty$, they are defined and are analytic for k in this domain.
2. $A(k)\overline{A(\bar{k})} - \rho B(k)\overline{B(\bar{k})} = 1$, $k \in \mathbb{C}$.
3. $A(k) = 1 + O\left(\frac{1+e^{2if_2(k)T}}{k}\right)$, $B(k) = O\left(\frac{e^{2if_2(k)T}}{k}\right)$, $k \to \infty$.
4. The inverse of the map (16.131) denoted by $\tilde{\mathbb{Q}}$ can be defined for the mKdVII, KdVII, and sG, respectively, as follows:

$$\begin{cases} g_0(t) = -2i\lim_{k\to\infty}\left(kM^{(t)}\right)_{12}, \\ g_1(t) = 4\lim_{k\to\infty}\left(k^2M^{(t)}\right)_{12} - 2ig_0(t)\lim_{k\to\infty}\left(kM^{(t)}\right)_{22}, \\ g_2(t) = 8i\lim_{k\to\infty}\left(k^3M^t\right)_{12} + \lambda(g_0(t))^3 + 4g_0(t)\lim_{k\to\infty}\left(k^2M^t\right)_{22} - 2ig_1(t)\lim_{k\to\infty}\left(kM^{(t)}\right)_{22}. \end{cases}$$

$$\begin{cases} g_0(t) = 4\lim_{k\to\infty}\left(k^2M^{(t)}\right)_{12}, \\ g_1(t) = 2i\lim_{k\to\infty}\left[4\left(k^3M^{(t)}\right)_{12} - g_0(t)kM^{(t)}_{22} - 4kg_0\right], \\ g_2(t) = g_0(t) + 2g_0(t)^2 + 2i\frac{d}{dt}\lim_{k\to\infty}\left(kM^{(t)}\right)_{11}. \end{cases}$$

$$\begin{cases} \cos g_0(t) = 1 - 2\lim_{k\to\infty}\left\{(kM^{(t)}(t,k))^2_{12} + 2i\partial_t(kM^{(t)}(t,k))_{22}\right\} \\ g_1(t) = -\frac{d}{dt}g_0(t) - 2\lim_{k\to\infty}(kM^{(t)}(t,k))_{12}. \end{cases} \quad (16.135)$$

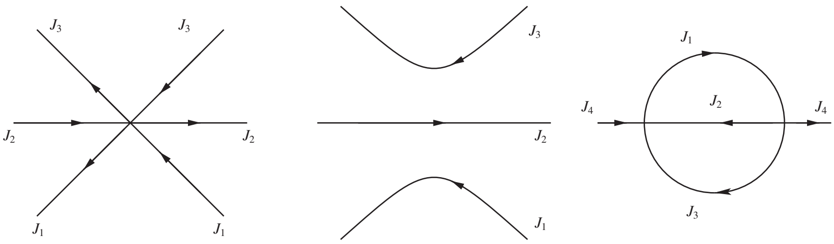


Figure 16.4. *The oriented contours $\mathcal{L}$ and the jump matrices J for the mKdVII, KdVII, and sG equations.*

In the above formulae, $M^{(t)}(t,k)$ is the unique solution of the following RH problem:

- $$M^{(t)}(t,k) = \begin{cases} M_+^{(t)}(t,k), & k \in D_1 \cup D_3, \\ M_-^{(t)}(t,k), & k \in D_2 \cup D_4, \end{cases}$$
  is a meromorphic function of k for $k \in \mathbb{C}\backslash\mathcal{L}$ and $\mathcal{L}$ is defined in section 16.2.1; see Figure 16.4.

- $$M^{(t)}(t,k) = I + O\left(\frac{1}{k}\right), \quad k \to \infty,$$
  $$M_-^{(t)}(t,k) = M_+^{(t)}(t,k) J^{(t)}(t,k), \quad k \in \mathcal{L},$$
  where
  $$J^{(t)}(t,k) = \begin{pmatrix} 1 & -\frac{B(k)}{\overline{A(\bar{k})}} e^{-2if_2(k)t} \\ \frac{\rho \overline{B(\bar{k})}}{A(k)} e^{2if_2(k)t} & \frac{1}{A(k)\overline{A(\bar{k})}} \end{pmatrix}. \tag{16.136}$$

- M satisfies appropriate residue conditions if $A(k)$ has zeros for $k \in D_1 \cup D_3$.

5. $\tilde{\mathbb{S}}^{-1} = \tilde{\mathbb{Q}}$.

6. For the KdVII,
$$A(k) = \frac{i\beta}{k} + O(1), \quad B(k) = -\frac{i\beta}{k} + O(1), \quad k \to 0, \tag{16.137}$$
where β is a real constant.

Proof. The derivation of these results, except for property 6, is similar to the derivation of the analogous results in Proposition 16.2. Property 6 is derived in the appendix of this chapter. □

Remark 16.9. If $T = \infty$, the functions $\{g_j(t)\}_{j=0}^{n-1}$ are assumed to belong to $S(\mathbb{R}_+)$, and the global relation takes the form

$$a(k)B(k) - b(k)A(k) = 0, \quad k \in \overline{D_1}. \tag{16.138}$$

Theorem 16.2. For the mKdVII and the KdVII let $q_0(x) \in S(\mathbb{R}^+)$, and for the sG let $q_0(x) - 2\pi m \in S(\mathbb{R}^+)$ and $q_1(x) \in S(\mathbb{R}^+)$, where m is an integer. Given these functions define $\{a(k), b(k)\}$ according to Definition 16.3. Suppose that there exist smooth functions $\{g_l(t)\}_0^{n-1}$ satisfying $\{g_l(0) = \partial_x^l q_0(0)\}_0^{n-1}$ such that the functions $\{A(k), B(k)\}$, which are defined from $g_l(t)$ according to Definition 16.4, satisfy the global relation (16.122), where $c(k)$ is analytic and bounded for $k \in D_1 \cup D_2$ and is of order $O(1/k)$, $k \to \infty$.

Define $M(x,t,k)$ as the solution of the following 2×2 matrix RH problem:

- M is meromorphic for k in $\mathbb{C}\backslash\mathcal{L}$, where $\mathcal{L}$ is defined as in section 16.2.1; see Figure 16.4.
- $$M_-(x,t,k) = M_+(x,t,k)J(x,t,k), \quad k \in \mathcal{L}, \tag{16.139}$$
  where M is M_- for $k \in D_2 \cup D_4$, M is M_+ for $k \in D_1 \cup D_3$, and J is defined in terms of a, b, A, B in (16.116).
- $$M(x,t,k) = I + O\left(\frac{1}{k}\right), \quad k \to \infty. \tag{16.140}$$
- M satisfies appropriate residue conditions if $a(k)$ has zeros in $D_1 \cup D_2$ and/or $d(k)$ has zeros in D_2.
- In the case of the KdVII, $M(x,t,k)$ has a pole at $k = 0$ satisfying
  $$M(x,t,k) \sim \frac{i\alpha(x,t)}{k}\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}, \quad k \to 0. \tag{16.141}$$

Then $M(x,t,k)$ exists and is unique.

Define $q(x,t)$ for the mKdVII, KdVII, sG, respectively, by

$$q(x,t) = -2i \lim_{k\to\infty} \left(kM(x,t,k)\right)_{12},$$

$$q(x,t) = -2i \lim_{k\to\infty} \partial_x\left(kM(x,t,k)\right)_{22},$$

$$\cos q(x,t) = 1 + 2\lim_{k\to\infty}\left\{\left(kM(x,t,k)\right)_{12}^2 + 2i\partial_x\left(kM(x,t,k)\right)_{22}\right\}. \tag{16.142}$$

Then $q(x,t)$ solves the mKdVII, the KdVII, and the sG, respectively. Furthermore

$$q(x,0) = q_0(x), \quad \{\partial_x^l q(0,t) = g_l(t)\}_0^{n-1}, \tag{16.143}$$

where $n = 2$ for the sG, and $n = 3$ for the mKdVII and KdVII. Also for the sG, $q_t(x,0) = q_1(x)$.

Proof. The proof is similar to that of Theorem 16.1. ∎

Remark 16.10. The proof that the solution $q(x,t)$ for $0 < t < T_*$, where $0 < T_* < T$, depends only on the boundary values for t between 0 and T_*, is given in [51].

Appendix A

A.1. The Eigenfunctions Associated with the sG Equation as $k \to 0$

The functions $\{a(k), b(k)\}$ are defined in terms of $\varphi(0,k)$; see (16.125). The vector $\varphi(x,k)$ is the second column vector of the matrix $\mu_3(x,k) = (\varphi^*(x,k), \varphi(x,k))$ which satisfies the following ODE:

$$\mu_x(x,k) + if_1(k)\hat{\sigma}_3\mu(x,k) = Q(x,k)\mu(x,k).$$

We will show that

$$\mu_3(x,k) = (-1)^m \left[\cos\left(\frac{q_0(x)}{2}\right) I - i\sin\left(\frac{q_0(x)}{2}\right)\sigma_1 + o(1)\right], \quad k \to 0, \tag{A.1}$$

where $q_0(x) \to 2\pi m$ as $x \to \infty$ and $I = \text{diag}(1,1)$.

Indeed, let

$$\mu_3(x,k) = \psi(x,k)E(x,k), \quad E(x,k) = e^{\frac{i}{4}x(k-\frac{1}{k})\sigma_3}.$$

Then $\psi(x,k)$ satisfies

$$\psi_x + \frac{i}{4}\left(k - \frac{1}{k}\right)\sigma_3\psi = Q(x,k), \quad \lim_{x\to\infty} \psi(x,k)E(x,k) = I,$$

where $Q(x,k)$ is defined by the third of equations (16.127). Thus

$$\psi_x = \frac{i}{4k}\left[\cos(q_0(x))\sigma_3 - \sin(q_0(x))\sigma_2\right]\psi + O(1), \quad k \to 0.$$

Noting that

$$\cos(q_0(x))\sigma_3 - \sin(q_0(x))\sigma_2 = f\sigma_3 f^{-1}, \quad f = \cos\left(\frac{q_0(x)}{2}\right) I - i\sin\left(\frac{q_0(x)}{2}\right)\sigma_1,$$

it follows that

$$\left(f^{-1}\psi\right)_x = \frac{i}{4k}\sigma_3\left(f^{-1}\psi\right) + O(1), \quad k \to 0.$$

Solving this equation and using the boundary condition

$$f^{-1}\psi \to (-1)^m \exp\left[\frac{i}{4k}x\sigma_3\right],$$

we find

$$\psi(x,k) = (-1)^m \left[\cos\left(\frac{q_0(x)}{2}\right) I - i\sin\left(\frac{q_0(x)}{2}\right)\sigma_1 + o(1)\right] e^{\frac{i}{4k}x\sigma_3}, \quad k \to 0,$$

which yields (A.1).

The functions $\{A(k), B(k)\}$ are defined in terms of $\Phi(T,k)$; see (16.132). The vector $\Phi(T,k)$ is the second column vector of the matrix $\mu_2(t,k) = (\Phi^*(t,k), \Phi(t,k))$, which satisfies the ODE

$$\mu_t(t,k) + if_2(k)\mu(t,k) = \tilde{Q}(t,k)\mu(t,k).$$

We shall show that

$$\mu_2(t,k) = \left[\cos\left(\frac{g_0(t)}{2}\right) I - i\sin\left(\frac{g_0(t)}{2}\right)\sigma_1 + o(1)\right] e^{-\frac{i}{4k}t\hat{\sigma}_3}\tilde{f}_0^{-1}, \quad k \to 0,$$

$$\tilde{f}_0 = \cos\left(\frac{g_0(0)}{2}\right) I - i\sin\left(\frac{g_0(0)}{2}\right)\sigma_1. \tag{A.2}$$

Indeed, let

$$\mu_2(t,k) = \Psi(t,k)\tilde{E}(t,k), \quad \tilde{E}(t,k) = e^{\frac{i}{4}(k+\frac{1}{k})t\sigma_3}.$$

Then $\Psi(t,k)$ satisfies

$$\Psi_t + \frac{i}{4}\left(k + \frac{1}{k}\right)\sigma_3\Psi = \tilde{Q}(t,k), \quad \Psi(0,k) = I,$$

where $\tilde{Q}(t,k)$ is defined by the third of equations (16.134). Thus,

$$\Psi_t = \frac{i}{4k}[\cos(g_0(t))\sigma_3 - \sin(g_0(t))\sigma_2]\Psi + O(1), \quad k \to 0.$$

The bracket in the above equation can be written as $\tilde{f}\sigma_3\tilde{f}^{-1}$, where $\tilde{f}$ is defined by

$$\tilde{f} = \cos\left(\frac{g_0(t)}{2}\right) I - i\sin\left(\frac{g_0(t)}{2}\right)\sigma_1;$$

thus

$$\left(\tilde{f}^{-1}\Psi\right)_t = \frac{i}{4k}\sigma_3\left(\tilde{f}^{-1}\Psi\right) + O(1), \quad k \to 0.$$

This equation, together with the boundary condition $(\tilde{f}^{-1}\Psi)(0,k) = \tilde{f}_0^{-1}$, yields (A.2).

A.2. The Eigenfunctions Associated with the KdVII Equation as $k \to 0$

Let $\mu(x,t,k)$ satisfy (16.92), where $f_1(k)$, $f_2(k)$, $Q(x,t,k)$, $\tilde{Q}(x,t,k)$ are defined by (16.94). Then

$$\mu(x,t,k) = i\frac{\alpha(x,t)}{k}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} + O(1), \quad k \to 0, \ \alpha(x,t) \text{ real}. \tag{A.3}$$

Indeed, the coefficient of $1/k$ in both equations (16.92) involves the matrix $\sigma_2 - i\sigma_3$. This suggests that

$$\mu(x,t,k) = \frac{1}{k}\begin{pmatrix} \alpha_1(x,t) & \alpha_2(x,t) \\ -\alpha_1(x,t) & -\alpha_2(x,t) \end{pmatrix} + O(1), \quad k \to 0.$$

The symmetry condition with respect to $k \mapsto -k$ (see (16.108b)) implies that $\alpha_2(x,t) = \alpha_1(x,t)$. Furthermore, the symmetry conditions with respect to complex conjugation (see (16.108a)) imply that $\alpha_1(x,t)$ is purely imaginary.

Equation (A.3) suggests that

$$\mu_3(x,k) = i\frac{\alpha(x)}{k}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} + O(1), \quad k \to 0, \quad \alpha(x) \text{ real}, \tag{A.4}$$

and

$$\mu_2(t,k) = i\frac{\beta(t)}{k}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} + O(1), \quad k \to 0, \quad \beta(t) \text{ real}. \tag{A.5}$$

These equations can be rigorously justified using the associated linear integral equations. The evaluation of (A.4) and (A.5) at $x = 0$ and $t = T$ determines the behavior of $\{a(k), b(k)\}$ and of $\{A(k), B(k)\}$ as $k \to 0$, i.e., (16.130) and (16.137).

Chapter 17

Linearizable Boundary Conditions

It was shown in Chapter 16 that the function $q(x, t)$ defined by (16.34) solves the nonlinear Schrödinger (NLS) equation and also satisfies the conditions

$$q(x, 0) = q_0(x), \quad q(0, t) = g_0(t), \quad q_x(0, t) = g_1(t)$$

if and only if there exist functions $g_0(t)$ and $g_1(t)$ such that the functions $\{A(k), B(k)\}$ (see Definition 16.2) satisfy the global relation (16.36). This implies that in order to solve a concrete initial-boundary value for the NLS, we must use the global relation to determine the unknown boundary value. For example, in order to solve the Dirichlet problem, i.e., the problem with the conditions

$$q(x, 0) = q_0(x), \quad 0 < x < \infty; \quad q(0, t) = g_0(t), \quad 0 < t < T,$$

we must construct the Dirichlet to Neumann map; i.e., we must determine $q_x(0, t)$ in terms of $q_0(x)$ and $q_0(t)$.

Actually, since the formula for $q(x, t)$ does *not* involve $q_x(0, t)$ directly but involves only $\{A(k), B(k)\}$, it is natural to ask the following question: Is it possible, given $q_0(x)$ and $g_0(t)$, to solve the global relation for $A(k)$ and $B(k)$? We recall that for the linearized version of the NLS, i.e., for equation (72), for which the analogue of $\{A(k), B(k)\}$ is $\{1, \tilde{g}(k)\}$, the function $\tilde{g}(k)$ can indeed be determined *directly* in terms of $q_0(x)$ and $g_0(t)$ (without the need to determine $q_x(0, t)$). Unfortunately, for the NLS with arbitrary boundary conditions it is *not* possible to determine directly $A(k)$ and $B(k)$. The situation for the sine-Gordon (sG), Korteweg–de Vries (KdV), and modified KdV (mKdV) equations is similar.

The reason that the global relation *cannot* be solved directly for $\{A(k), B(k)\}$ is the following: For linear PDEs, the transforms of $\{\partial_x^j q(0, t)\}_0^{n-1}$, denoted by $\tilde{g}_j(w(k))$, remain invariant under those transformations $k \to \nu(k)$ which leave $w(k)$ invariant. However, Definition 16.4 implies that the functions $\{A(k), B(k)\}$, in addition to the $\exp[if_2(k)t]$ (which remains invariant under $k \to \nu(k)$), also involve the functions $(\Phi_1(t, k), \Phi_2(t, k))$ which in general do *not* possess any symmetry properties with respect to the transformation $k \to \nu(k)$. However, for a *particular* class of boundary conditions, the functions $(\Phi_1(t, k), \Phi_2(t, k))$ *do* possess such symmetry properties, in which case it is possible to solve directly the global relation for $\{A(k), B(k)\}$.

In what follows we present a general method for identifying, as well as analyzing, this class of boundary value problems, which will be referred to as *linearizable*.

Proposition 17.1. Suppose that the t-part of the Lax pair of an integrable nonlinear PDE is characterized by the scalar function $f_2(k)$ and by the 2×2 matrix-valued function $\tilde{Q}(x,t,k)$; see (16.92b). Let $k \to \nu(k)$ be the transformation in the complex k-plane which leaves $f_2(k)$ invariant, i.e.,

$$f_2(\nu(k)) = f_2(k), \quad \nu(k) \neq k. \tag{17.1}$$

Let $U(t,k)$ be defined in terms of f_2 and $\tilde{Q}$ by

$$U(t,k) = if_2(k)\sigma_3 - \tilde{Q}(0,t,k). \tag{17.2}$$

Suppose that it is possible to compute explicitly, in terms of the given boundary conditions, a nonsingular matrix $N(k)$ such that

$$U(t,\nu(k))N(k) = N(k)U(t,k). \tag{17.3}$$

Then the functions $\{A(k), B(k)\}$ defined by Definition 16.4 possess the following symmetry properties:

$$\begin{aligned}\overline{P(\bar{k})}A(\nu(k)) = {} & \overline{N_1(\bar{k})}\,\overline{N_4(\bar{k})}A(k) + \rho\overline{N_2(\bar{k})}\,\overline{N_4(\bar{k})}B(k) \\ & -e^{2if_2(k)T}\overline{N_3(\bar{k})}\left[\overline{N_2(\bar{k})}\,\overline{A(\bar{k})} + \overline{N_1(\bar{k})}\,\overline{B(\bar{k})}\right],\end{aligned} \tag{17.4a}$$

$$\begin{aligned}P(k)B(\nu(k)) = {} & N_1(k)N_2(k)A(k) + N_1(k)^2B(k) \\ & -e^{2if_2(k)T}N_2(k)\left[N_1(k)\overline{A(\bar{k})} + \rho N_2(k)\overline{B(\bar{k})}\right],\end{aligned} \tag{17.4b}$$

where

$$N_1 = N_{11}, \quad N_2 = N_{12}, \quad N_3 = N_{21}, \quad N_4 = N_{23}, \quad P = N_1N_4 - N_2N_3. \tag{17.5}$$

Proof. Recall that

$$\Phi_1(T,k) = -e^{-2if_2(k)T}B(k), \quad \Phi_2(T,k) = \overline{A(\bar{k})}. \tag{17.6}$$

Define $\mu_2(t,k)$ by

$$\mu_2(t,k) = \begin{pmatrix} \overline{\Phi_2(t,\bar{k})} & \Phi_1(t,k) \\ \rho\overline{\Phi_1(t,\bar{k})} & \Phi_2(t,k) \end{pmatrix}.$$

Then μ_2 satisfies

$$\partial_t\mu_2(t,k) + if_2(k)\hat{\sigma}_3\mu_2(t,k) = \tilde{Q}(0,t,k)\mu_2(t,k),$$

$$\mu_2(0,k) = I.$$

Define M_1 and M_2 by

$$M_1(t,k) = \Phi_1(t,k)e^{if_2(k)t}, \quad M_2(t,k) = \Phi_2(t,k)e^{if_2(k)t}.$$

Then the matrix M, defined by

$$M(t,k) = \begin{pmatrix} \overline{M_2(t,\bar{k})} & M_1(t,k) \\ \rho\overline{M_1(t,\bar{k})} & M_2(t,k) \end{pmatrix},$$

satisfies the following equations:

$$\partial_t M(t,k) + if_2(k)\sigma_3 M(t,k) = \tilde{Q}(0,t,k)M(t,k), \tag{17.7a}$$

$$M(0,k) = I. \tag{17.7b}$$

Furthermore, (17.6) and the definition of M_1 and M_2 imply

$$M_1(T,k) = -e^{-if_2(k)T}B(k), \quad M_2(T,k) = \overline{A(\bar{k})}e^{if_2(k)T}. \tag{17.8}$$

If the matrix $U(t,k)$, defined by (17.2), satisfies the symmetry property (17.3), then $M(t,k)$ satisfies the following symmetry properties:

$$M(t,\nu(k)) = N(k)M(t,k)N(k)^{-1}. \tag{17.9}$$

Indeed, (17.7a) implies (17.9) to within a multiplicative nonsingular matrix, and this matrix equals I because of (17.7b).

The second column vector of (17.9) evaluated at $t = T$ implies the following equations:

$$P(k)M_1(T,\nu(k)) = -N_1N_2\overline{M_2} + N_1^2M_1 - \rho N_2^2\overline{M_1} + N_1N_2M_2, \tag{17.10a}$$

$$P(k)M_2(T,\nu(k)) = -N_2N_3\overline{M_2} + N_1N_3M_1 - \rho N_2N_4\overline{M_1} + N_1N_4M_2, \tag{17.10b}$$

where we have used the notation $N_1N_2\overline{M_2} = N_1(k)N_2(k)\overline{M_2(T,\bar{k})}$, etc. Taking the Schwarz conjugate of (17.10b) and then expressing, in the resulting equation and in (17.10a), M_1 and M_2 in terms of A and B using (17.8), we find (17.4). □

Remark 17.1. By taking the determinant of (17.3) it follows that a *necessary* condition for the existence of linearizable boundary conditions is that the k-dependence of the determinant of the matrix $U(t,k)$ is of the form $f_2(k)$.

Example 17.1 (NLS). In the case of the NLS the matrix $U(t,k)$ is given by

$$U(t,k) = \begin{pmatrix} 2ik^2 + i\lambda|q(0,t)|^2 & -2kq(0,t) - iq_x(0,t) \\ -2\lambda k\bar{q}(0,t) + i\lambda\overline{q_x}(0,t) & -2ik^2 - i\lambda|q(0,t)|^2 \end{pmatrix}. \tag{17.11}$$

Thus, the determinant of $U(t,k)$ depends on k in the form of k^2, provided that

$$q(0,t)\bar{q}_x(0,t) - \bar{q}(0,t)q_x(0,t) = 0. \tag{17.12}$$

If this condition is satisfied, (17.3) yields

$$(2kq - iq_x)N_3 = -\lambda(2k\bar{q} - i\bar{q}_x)N_2, \tag{17.13a}$$

$$(2kq + iq_x)N_1 + (2kq - iq_x)N_4 = -2(2ik^2 + i\lambda|q|^2)N_2. \tag{17.13b}$$

We now discuss in detail three particular solutions of (17.12).

(a) $q(0,t) = 0$. In this case $\tilde{Q}(0,t,k)$ is a function of t and k^2, thus $M(t,k) = M(t,-k)$ and there is no need to introduce $N(k)$, i.e., $N(k) = I$. Then equations (17.4), with $N_1 = N_4 = 1$, $N_2 = N_3 = 0$, yield

$$A(-k) = A(k), \quad B(-k) = B(k), \quad k \in \mathbb{C}. \tag{17.14}$$

(b) $q_x(0,t) = 0$. Equations (17.13) imply that $N(k)$ does not depend on $q(0,t)$, provided that $N_2 = N_3 = 0$ and $N_4 = -N_1$. Then equations (17.4) yield

$$A(-k) = A(k), \quad B(-k) = -B(k), \quad k \in \mathbb{C}. \tag{17.15}$$

(c) $q_x(0,t) - \chi q(0,t) = 0$, χ positive constant. Equations (17.13) imply that $N(k)$ does not depend separately on $q(0,t)$ and $q_x(0,t)$ provided that $N_2 = N_3 = 0$ and

$$(2k - i\chi)N_4 + (2k + i\chi)N_1 = 0.$$

Then equations (17.4) yield

$$A(-k) = A(k), \quad B(-k) = -\frac{2k - i\chi}{2k + i\chi}B(k), \quad k \in \mathbb{C}. \tag{17.16}$$

Using the transformations (17.14)–(17.16), together with the global relation, it is possible to express $A(k)$ and $B(k)$ in terms of $a(k)$ and $b(k)$.

Remark 17.2 If $2k + i\chi = 0$, the equation coupling N_1 with N_4 shows that $N_4(-i\chi/2) = 0$ and then (17.4b) implies $B(-i\chi/2) = 0$, thus $k = -i\chi/2$ is a removable singularity. Similar considerations are valid for Examples 7.2 and 7.3, see [178].

For convenience we assume $T = \infty$. It can be shown that a similar analysis is valid if $T < \infty$. If $T = \infty$, the global relation becomes

$$a(k)B(k) - b(k)A(k) = 0, \quad \arg k \in [0, \pi/2]. \tag{17.17}$$

We note that since $T = \infty$, $A(k)$ and $B(k)$ are *not* entire functions but are defined only for

$$\arg k \in [0, \pi/2] \cup [\pi, 3\pi/2].$$

(a) $q(0,t)=0$. Letting $k\mapsto -k$ in the definition of $\overline{d(\bar{k})}$ and using the symmetry relations (17.14), we find

$$A(k)\overline{a(-\bar{k})}-\lambda B(k)\overline{b(-\bar{k})}=\overline{d(-\bar{k})},\quad \arg k\in[0,\pi/2]\,. \tag{17.18}$$

This equation and the global relation (17.17) are *two algebraic* equations for $A(k)$ and $B(k)$. Their solution yields

$$A(k)=\frac{a(k)\overline{d(-\bar{k})}}{\Delta_0(k)},\quad B(k)=\frac{b(k)\overline{d(-\bar{k})}}{\Delta_0(k)},\quad \arg k\in[0,\pi/2]\,, \tag{17.19}$$

where $\Delta_0(k)$ is defined by

$$\Delta_0(k)=a(k)\overline{a(-\bar{k})}-\lambda b(k)\overline{b(-\bar{k})}. \tag{17.20}$$

The function $\overline{d(\bar{k})}$ can be computed explicitly in terms of $a(k)$ and $b(k)$. However, it does not affect the solution of the RH problem of Theorem 16.1. Indeed, this RH problem is defined in terms of $\gamma(k)=b(k)/\bar{a}(k)$, $k\in\mathbb{R}$, and of $\Gamma(k)$ which involves $a(k)$, $b(k)$, and $A(k)/B(k)$. Using (17.19) we find the following expression for $\Gamma(k)$:

$$\Gamma(k)=\frac{\lambda\left(\overline{B(\bar{k})}/\overline{A(\bar{k})}\right)}{a(k)\left(a(k)-\lambda b(k)\left(\overline{B(\bar{k})}/\overline{A(\bar{k})}\right)\right)}=\frac{\lambda\overline{b(-\bar{k})}}{a(k)\Delta_0(k)},\quad k\in\mathbb{R}^-\cup i\mathbb{R}^+. \tag{17.21}$$

The function $\Delta_0(k)$ is an analytic function in the upper half k-plane and it satisfies the symmetry equation

$$\Delta_0(k)=\overline{\Delta_0(-\bar{k})}. \tag{17.22}$$

It can be shown that the zero set of $\Delta_0(k)$ is the union of the following zeros:

$$\{\lambda_j\}_{j=1}^{\Lambda}\cup\{-\bar{\lambda}_j\}_{j=1}^{\Lambda}. \tag{17.23}$$

where $\{\lambda_j\}_1^{\Lambda}$, $\arg\lambda_j\in(\frac{\pi}{2},\pi)$, $j=1,\dots,\Lambda$ are the zeros of $d(k)$. Indeed, the global relation (17.17) implies that the zero sets of the functions $a(k)$ and $A(k)$ coincide in the first quadrant. It also implies that if the zeros of $a(k)$ are simple, the zeros of $A(k)$ have the same property. This fact together with (17.19) imply that the zero sets of the functions $\overline{d(-\bar{k})}$ and $\Delta_0(k)$ coincide in the first quadrant as well. Equation (17.22) implies that the zero set of $\Delta_0(k)$ is the set given in (17.23).

Since the zeros λ_j of $d(k)$ coincide with the second quadrant zeros of $\Delta_0(k)$, (17.19) and (17.14) imply the following modification of the residue conditions:

$$\underset{k_j}{\text{Res}}\,[M(x,t,k)]_1=\frac{1}{\dot{a}(k_j)b(k_j)}e^{2i\theta(k_j)}[M(x,t,k_j)]_2,\quad j=1,\dots,n_1, \tag{17.24a}$$

$$\underset{\bar{k}_j}{\text{Res}}\,[M(x,t,k)]_2=\frac{\lambda}{\bar{\dot{a}}(k_j)\bar{b}(k_j)}e^{-2i\theta(\bar{k}_j)}[M(x,t,\bar{k}_j)]_1,\quad j=1,\dots,n_1, \tag{17.24b}$$

$$\underset{\lambda_j}{\text{Res}}\,[M(x,t,k)]_1=\frac{\lambda\overline{b(-\bar{\lambda}_j)}}{a(\lambda_j)\dot{\Delta}_0(\lambda_j)}e^{2i\theta(\lambda_j)}[M(x,t,\lambda_j)]_2,\quad j=1,\dots,\Lambda, \tag{17.24c}$$

$$\operatorname*{Res}_{\bar{\lambda}_j}[M(x,t,k)]_2 = \frac{b(-\bar{\lambda}_j)}{\overline{a}(\lambda_j)\dot{\overline{\Delta}}_0(\lambda_j)} e^{-2i\theta(\bar{\lambda}_j)}[M(x,t,\bar{\lambda}_j)]_1, \quad j=1,\ldots,\Lambda, \tag{17.24d}$$

where

$$\theta(k_j) = k_j x + 2k_j^2 t. \tag{17.25}$$

(b) $q_x(0,t)=0$. Proceeding as in (a) we find that $\{A, B\}$ are given by (17.19), where $\Delta_0(k)$ is replaced by

$$\Delta_1(k) = a(k)\overline{a(-\bar{k})} + \lambda b(k)\overline{b(-\bar{k})}. \tag{17.26}$$

The zeros λ_j are now the second quadrant zeros of $\Delta_1(k)$, and (17.21) is now replaced by

$$\Gamma(k) = -\frac{\lambda\overline{b(-\bar{k})}}{a(k)\Delta_1(k)}, \quad k \in \mathbb{R}^- \cup i\mathbb{R}^+. \tag{17.27}$$

The modified residue conditions are given by the following equations:

$$\operatorname*{Res}_{k_j}[M(x,t,k)]_1 = \frac{1}{\dot{a}(k_j)b(k_j)} e^{2i\theta(k_j)}[M(x,t,k_j)]_2, \quad j=1,\ldots,n_1, \tag{17.28a}$$

$$\operatorname*{Res}_{\bar{k}_j}[M(x,t,k)]_2 = \frac{\lambda}{\dot{\bar{a}}(k_j)\bar{b}(k_j)} e^{-2i\theta(\bar{k}_j)}[M(x,t,\bar{k}_j)]_1, \quad j=1,\ldots,n_1, \tag{17.28b}$$

$$\operatorname*{Res}_{\lambda_j}[M(x,t,k)]_1 = -\frac{\lambda\overline{b(-\bar{\lambda}_j)}}{a(\lambda_j)\dot{\Delta}_1(\lambda_j)} e^{2i\theta(\lambda_j)}[M(x,t,\lambda_j)]_2, \quad j=1,\ldots,\Lambda, \tag{17.28c}$$

$$\operatorname*{Res}_{\bar{\lambda}_j}[M(x,t,k)]_2 = -\frac{b(-\bar{\lambda}_j)}{\overline{a}(\lambda_j)\dot{\overline{\Delta}}_1(\lambda_j)} e^{-2i\theta(\bar{\lambda}_j)}[M(x,t,\bar{\lambda}_j)]_1, \quad j=1,\ldots,\Lambda. \tag{17.28d}$$

(c) $q_x(0,t) - \chi q(0,t) = 0$, χ constant. In this case, $\Delta_0(k)$ is replaced by

$$\Delta_\chi(k) = a(k)\overline{a(-\bar{k})} + \lambda\frac{2k-i\chi}{2k+i\chi} b(k)\overline{b(-\bar{k})}. \tag{17.29}$$

The zeros λ_j are now the second quadrant zeros of $\Delta_\chi(k)$, and

$$\Gamma(k) = -\frac{\lambda\frac{2k-i\chi}{2k+i\chi}\overline{b(-\bar{k})}}{a(k)\Delta_\chi(k)}, \quad k \in \mathbb{R}^- \cup i\mathbb{R}^+. \tag{17.30}$$

The modified residue conditions are given by the following equations:

$$\operatorname*{Res}_{k_j}[M(x,t,k)]_1 = \frac{1}{\dot{a}(k_j)b(k_j)} e^{2i\theta(k_j)}[M(x,t,k_j)]_2, \quad j=1,\ldots,n_1, \tag{17.31a}$$

$$\operatorname*{Res}_{\bar{k}_j}[M(x,t,k)]_2 = \frac{\lambda}{\dot{\bar{a}}(k_j)\bar{b}(k_j)} e^{-2i\theta(\bar{k}_j)}[M(x,t,\bar{k}_j)]_1, \quad j=1,\ldots,n_1, \tag{17.31b}$$

$$\operatorname*{Res}_{\lambda_j}[M(x,t,k)]_1 = -\frac{\lambda\frac{2\lambda_j-i\chi}{2\lambda_j+i\chi}\overline{b(-\bar{\lambda}_j)}}{a(\lambda_j)\dot{\Delta}_\chi(\lambda_j)} e^{2i\theta(\lambda_j)}[M(x,t,\lambda_j)]_2, \quad j=1,\ldots,\Lambda, \tag{17.31c}$$

$$\underset{\bar{\lambda}_j}{\text{Res}}\,[M(x,t,k)]_2 = -\frac{\frac{2\bar{\lambda}_j+i\chi}{2\bar{\lambda}_j-i\chi}b(-\bar{\lambda}_j)}{\overline{a}(\lambda_j)\dot{\overline{\Delta}}_\chi(\lambda_j)}e^{-2i\theta(\bar{\lambda}_j)}[M(x,t,\bar{\lambda}_j)]_1, \quad j=1,\ldots,\Lambda. \quad (17.31\text{d})$$

Theorem 16.1 and the above results imply the following.

Let $q(x,t)$ satisfy the NLS equation, the initial condition

$$q(x,0) = q_0 \in S(\mathbb{R}^+), \quad 0 < x < \infty,$$

and any one of the following boundary conditions:

(a) $q(0,t) = 0, \quad t > 0,$

or

(b) $q_x(0,t) = 0, \quad t > 0,$

or

(c) $q_x(0,t) - \chi q(0,t) = 0, \quad \chi > 0, \quad t > 0.$

Assume that the initial and boundary conditions are compatible at $x = t = 0$. Furthermore, assume the following.

(i) $a(k)$, which is defined in Definition 16.1, can have at most a finite number of simple zeros for $\text{Im}\,k > 0$.

(ii) $\Delta_0(k)$ in case (a), or $\Delta_1(k)$ in case (b), or $\Delta_\chi(k)$ in case (c), can have at most a finite number of simple zeros in the second quadrant which do not coincide with the possible zeros of $a(k)$ (Δ_0, Δ_1, Δ_χ are defined in (17.20), (17.26), (17.29)).

The solution $q(x,t)$ can be constructed through (16.34), where M satisfies the RH problem defined in Theorem 16.1, with $\Gamma(k)$ given by (17.21) in case (a), by (17.27) in case (b), and by (17.30) in case (c). The relevant residue conditions are given by (17.24) in case (a), by (17.28) in case (b), and by (17.31) in case (c).

Example 17.2 (sG). The determinant of the matrix U is proportional to

$$\left(k+\frac{1}{k}\right)^2 + 2\left[\cos q(0,t) - 1\right] + \left(\frac{dq(0,t)}{dt} + q_x(0,t)\right)^2.$$

Thus the necessary condition for linearizable boundary conditions is *always* satisfied. We consider the case of

$$q(0,t) = \chi, \quad \chi \text{ real constant}. \quad (17.32)$$

The invariance of $f_2(k)$ yields

$$\nu(k) = \frac{1}{k}.$$

Letting $N_3 = N_2$, $N_4 = N_1$, we find that both the (11) and the (22) elements of (17.3) yield (17.33a) below, while both the (12) and (21) elements of (17.3) yield (17.39b) below:

$$(k^2-1)(\cos\chi - 1)N_1 = i(k^2+1)(\sin\chi)N_2, \quad (17.33\text{a})$$

$$i(k^2-1)(\sin\chi)N_1 = (k^2+1)(\cos\chi+1)N_2. \tag{17.33b}$$

These equations are clearly equivalent. Using either of these equations, we find

$$N_1 = fN_2, \quad N_3 = N_2, \quad N_4 = fN_2,$$

where

$$f(k) = i\frac{k^2+1}{k^2-1}\frac{\sin\chi}{\cos\chi-1}. \tag{17.34}$$

Then, using $\overline{f(\bar{k})} = \rho f(k)$, $\rho = -1$, equations (17.4) yield

$$(f^2-1)B\left(\frac{1}{k}\right) = f\Big[A(k)+fB(k)\Big] - e^{2if_2(k)T}\Big[f\overline{A(\bar{k})}+\rho\overline{B(\bar{k})}\Big],$$

$$(f^2-1)A\left(\frac{1}{k}\right) = f\Big[fA(k)+B(k)\Big] - e^{2if_2(k)T}\Big[\overline{A(\bar{k})}+\rho f\overline{B(\bar{k})}\Big]. \tag{17.35}$$

In order to simplify these equations we first take their Schwarz conjugate,

$$(f^2-1)\overline{B\left(\frac{1}{\bar{k}}\right)} = f\Big[\rho\overline{A(\bar{k})}+f\overline{B(\bar{k})}\Big] - \rho e^{-2if_2(k)T}\Big[fA(k)+B(k)\Big],$$

$$(f^2-1)\overline{A\left(\frac{1}{\bar{k}}\right)} = f\Big[f\overline{A(\bar{k})}+\rho\overline{B(\bar{k})}\Big] - e^{-2if_2(k)T}\Big[A(k)+fB(k)\Big]. \tag{17.36}$$

Eliminating from (17.35a) and (17.36b) the term $f\overline{A(\bar{k})}+\rho\overline{B(\bar{k})}$, we find

$$B\left(\frac{1}{k}\right) = -\frac{e^{2if_2T}}{f}\overline{A\left(\frac{1}{\bar{k}}\right)} + \frac{A(k)}{f} + B(k). \tag{17.37a}$$

Similarly, eliminating from (17.35b) and (17.36a) the term $\overline{A(\bar{k})}+\rho f\overline{B(\bar{k})}$, we find

$$A\left(\frac{1}{k}\right) = -\rho\frac{e^{2if_2T}}{f}\overline{B\left(\frac{1}{\bar{k}}\right)} + A(k) + \frac{B(k)}{f}. \tag{17.37b}$$

Letting $k\to\frac{1}{k}$ in the definition of $\overline{d(\bar{k})}$ and then replacing $A(\frac{1}{k})$ and $B(\frac{1}{k})$ in the resulting expression by the RHS of equations (17.37), we find

$$\left[\overline{a\left(\frac{1}{\bar{k}}\right)} - \frac{\rho}{f}\overline{b\left(\frac{1}{\bar{k}}\right)}\right]A(k) + \left[\rho\overline{b\left(\frac{1}{\bar{k}}\right)} + \frac{1}{f}\overline{a\left(\frac{1}{\bar{k}}\right)}\right]B(k) = \overline{d\left(\frac{1}{\bar{k}}\right)} + \frac{\rho}{f}\overline{c\left(\frac{1}{\bar{k}}\right)},\ k\in D_1. \tag{17.38}$$

For the derivation of this equation we have used the remarkable fact that the terms $\exp[2if_2T]$, $\bar{A}$, and $\bar{B}$ are eliminated because of the global relation (16.122).

Solving (17.38) and the global relation (which is also valid for $k\in D_1$) for $A(k)$ and $B(k)$, we find

$$A(k) = \frac{1}{\Delta(k)}\left\{a(k)\left[\overline{d\left(\frac{1}{\bar{k}}\right)} - \frac{\rho}{f}\overline{c\left(\frac{1}{\bar{k}}\right)}\right] + e^{2if_2T}\left[\rho\overline{b\left(\frac{1}{\bar{k}}\right)} - \frac{1}{f}\overline{a\left(\frac{1}{\bar{k}}\right)}\right]c(k)\right\},$$

$$B(k) = \frac{1}{\Delta(k)}\left\{b(k)\left[\overline{d\left(\frac{1}{\bar{k}}\right)} - \frac{\rho}{f}\overline{c\left(\frac{1}{\bar{k}}\right)}\right] + e^{2if_2T}\left[\overline{a\left(\frac{1}{\bar{k}}\right)} - \frac{\rho}{f}\overline{b\left(\frac{1}{\bar{k}}\right)}\right]c(k)\right\}, \quad k \in D_1, \tag{17.39}$$

where

$$\Delta(k) = a(k)\overline{a\left(\frac{1}{\bar{k}}\right)} - \rho b(k)\overline{b\left(\frac{1}{\bar{k}}\right)} + \frac{1}{f}\left[-\rho a(k)\overline{b\left(\frac{1}{\bar{k}}\right)} + b(k)\overline{a\left(\frac{1}{\bar{k}}\right)}\right] \tag{17.40}$$

and $\rho = -1$.

In order to obtain $A(k)$ and $B(k)$ for $k \in D_3$, we let $k \to \frac{1}{k}$ in the global relation and then replace $A(1/k)$ and $B(1/k)$ in the resulting expression by the RHS of equations (17.37); this yields the following equation:

$$\left[\frac{1}{f}a\left(\frac{1}{k}\right) - b\left(\frac{1}{k}\right)\right]A(k) + \left[a\left(\frac{1}{k}\right) - \frac{1}{f}b\left(\frac{1}{k}\right)\right]B(k) = e^{2if_2T}\left[c\left(\frac{1}{k}\right) + \frac{1}{f}d\left(\frac{1}{k}\right)\right], \quad k \in D_3. \tag{17.41}$$

This time, the terms $\bar{A}$ and $\bar{B}$ are eliminated using the definition of $d(\frac{1}{k})$.

Solving (17.41) and the equation defining $\overline{d(\bar{k})}$ (which is also valid for $k \in D_3$) we find

$$A(k) = \frac{1}{\overline{\Delta(\bar{k})}}\left\{\left[a\left(\frac{1}{k}\right) - \frac{1}{f}b\left(\frac{1}{k}\right)\right]\overline{d(\bar{k})} + \rho e^{2if_2T}\overline{b(\bar{k})}\left[c\left(\frac{1}{k}\right) + \frac{1}{f}d\left(\frac{1}{k}\right)\right]\right\},$$

$$B(k) = \frac{1}{\overline{\Delta(\bar{k})}}\left\{\left[b\left(\frac{1}{k}\right) - \frac{1}{f}a\left(\frac{1}{k}\right)\right]\overline{d(\bar{k})} + e^{2if_2T}\overline{a(\bar{k})}\left[c\left(\frac{1}{k}\right) + \frac{1}{f}d\left(\frac{1}{k}\right)\right]\right\}, \quad k \in D_3. \tag{17.42}$$

Following arguments very similar to those used in Chapter 16 (see [51] for details), it can be shown that $A(k)$, $B(k)$ can be replaced by the expressions obtained from the RHS of (17.39) and (17.42) after deleting the terms involving $\exp(2if_2T)$. Then the ratio B/A yields

$$\frac{B(k)}{A(k)} = \frac{f(k)b\left(\frac{1}{k}\right) - a\left(\frac{1}{k}\right)}{f(k)a\left(\frac{1}{k}\right) - b\left(\frac{1}{k}\right)}, \quad k \in D_3. \tag{17.43}$$

In summary, the sG with the boundary condition $q(0,t) = \chi$, χ real constant, can be solved in terms of the RH problem defined in Theorem 16.2, where $\Gamma(k)$ is explicitly given in terms of $\{a(k), b(k), B(k)/A(k)\}$ by (16.118) and B/A is explicitly defined in terms of $\{a, b, f\}$ by (17.43), with $f(k)$ explicitly defined in terms of χ by (17.34).

The discrete spectrum can be analyzed using arguments similar to those used for the NLS.

Example 17.3 (KdV). The determinant of the matrix U equals

$$(k + 4k^3)^2 - \left\{q_x(0,t)^2 + [2 + 4q(0,t)]V(t) + 4k^2[(q(0,t))^2 + 2V(t)]\right\},$$

$$V(t) = [q(0,t)]^2 + \frac{1}{2}q(0,t) - \frac{1}{2}q_{xx}(0,t). \tag{17.44}$$

The condition that the coefficient of k^2 vanishes yields

$$3[q(0,t)]^2 + q(0,t) - q_{xx}(0,t) = 0. \tag{17.45}$$

The invariance of $f_2(k)$ yields

$$\nu^2 + k\nu + k^2 + \frac{1}{4} = 0. \tag{17.46}$$

We will consider two particular solutions of (17.45).

(a) $q(0,t) = q_{xx}(0,t) = 0$. In this case the matrix $if_2\sigma_3 - \tilde{Q}(0,t,k)$ depends on k only through $f_2(k)$, thus $N = I$, and

$$A(k) = A(\nu(k)), \quad B(k) = B(\nu(k)), \quad k \in \mathbb{C}. \tag{17.47}$$

These equations, following the arguments used in Example 17.1, imply

$$\Gamma(k) = \frac{\rho\overline{b(\nu(\bar{k}))}}{a(k)\Delta_0(k)}, \quad \Delta_0(k) = a(k)\overline{a(\nu(\bar{k})} - \rho b(k)\overline{b(\nu(\bar{k})}, \quad k \in D_2, \quad \rho = -1. \tag{17.48}$$

(b) $q(0,t) = \chi$, $q_{xx}(0,t) = \chi + 3\chi^2$, χ real constant, $\chi \neq 0$. Letting $N_3 = N_2$, $N_4 = N_1$, we find that both the (1-1) and the (2-2) elements of (17.3) yield (17.49a) below, while both the (1-2) and (2-1) elements of (17.3) yield (17.49b) below:

$$(\nu - k)VN_1 = (\nu + k)(V + 2\chi k\nu)N_2, \tag{17.49a}$$

$$(\nu - k)(V - 2\chi k\nu)N_1 = [V(\nu + k) - 2k\nu(k + 4k^3)]N_2. \tag{17.49b}$$

These two equations are equivalent; indeed, their ratio yields

$$\frac{k + 4k^3}{k + \nu} = \frac{2\chi^2}{V}k\nu,$$

and since the definition of V and the boundary conditions imply $V = -\chi^2/2$, the above equation becomes (17.46).

Since the form of N is the same as the one used in the sG, it follows that the derivation of B/A is identical to that for the sG, except that $1/k$ is now replaced by $\nu(k)$ defined by (17.46) and $f(k)$ is now defined by

$$f(k) = \frac{\nu + k}{\nu - k}\left(1 - \frac{4k\nu}{\chi}\right). \tag{17.50}$$

In summary, the KdVII with the boundary conditions $q(0,t) = q_{xx}(0,t) = 0$ can be solved in terms of the RH problem defined in Theorem 16.2, where $\Gamma(k)$ is explicitly given in terms of $\{a(k), b(k)\}$ by (17.48). Similarly, for the case of the boundary conditions (b) above, $\Gamma(k)$ is given by (16.118), where

$$\frac{B(k)}{A(k)} = \frac{f(k)b(\nu(k)) - a(\nu(k))}{f(k)a(\nu(k)) - b(\nu(k))}, \tag{17.51}$$

and $\nu(k)$, $f(k)$ are defined by (17.46) and (17.50).

17.1 Additional Linearizable Boundary Value Problems

A well-posed problem for the mKdVII equation (16.89) requires two boundary conditions at $x = 0$. In the particular case of $q(0,t) = q_x(0,t) = 0$, the matrix $\tilde{Q}(0,t,k)$ is independent of k, and thus this problem can be easily solved by taking $N = I$.

The sG equation with the boundary condition

$$q_x(0,t) + \chi_1 \cos\left(\frac{q(0,t)}{2}\right) + \chi_2 \sin\left(\frac{q(0,t)}{2}\right) = 0 \tag{17.52}$$

is solved in [56]. This case can be analyzed using the methodology developed in this chapter, but it requires the employment of a *different* Lax pair from the one used earlier.

The results presented in this chapter indicate that even for nonlinear PDEs with third order derivatives, it is possible to solve linearizable boundary value problems with the *same* level of efficiency as initial-value problems. Indeed, the relevant RH problem is formulated in terms of the explicit function $f(k)$ defined in terms of the boundary conditions and in terms of the functions $\{a(k), b(k)\}$ which are defined in terms of the initial condition $q_0(x)$, $x > 0$. The only difference in this RH problem from the one obtained for initial-value problems is that the relevant jumps occur on a more complicated contour than the real axis. But this difference does not affect at all the effectiveness of the solution.

Remark 17.2. Linearizable boundary value problems for the NLS and the sG equations have been studied via techniques based on an appropriate continuation of the half-line problem to the problem on the line (see [105], [106], [107], [108], [109]. The solutions are given via the RH problems corresponding to the extended initial-value problems. These continuations are described by explicit conditions on the scattering data associated with the initial-value problem on the line (see [105], [107], [108], [109]). In the case where either $q(0,t)$ or $q_x(0,t)$ vanishes, these conditions can be easily translated into the even or odd continuation of the initial data $q_0(x)$. Although a continuation is still possible for other boundary value problems such as the case (c) of Example 17.1, the relevant procedure is more complicated. Furthermore, what is more important, the procedure introduced here is also valid for PDEs involving *third* order derivatives; see Example 17.3.

Remark 17.3. Linearizable boundary value problems have infinitely many conserved quantities [110].

Remark 17.4. Explicit soliton solutions for linearizable cases involving the NLS and an integrable generalization of the NLS are presented in [23] and [178] respectively.

Chapter 18

The Generalized Dirichlet to Neumann Map

It was noted in Chapter 17 that, in general it is *not* possible to solve the global relation directly for $\{A(k), B(k)\}$. However, it will be shown here that it is possible to solve the global relation for the unknown boundary values, i.e., it is possible to characterize the generalized Dirichlet to Neumann map. The relevant methodology provides a nonlinear analogue of the approach introduced in Chapter 1, section 1.1 for the solution of the corresponding problem for linear evolution PDEs. Thus, in order to facilitate the understanding of the nonlinear methodology, we first review the construction of the Dirichlet to Neumann map for the linearized version of the nonlinear Schrödinger (NLS) equation, i.e., for (72).

We consider (72) with $q(x,0) = q_0(x)$ and $q(0,t) = g_0(t)$, and our goal is to determine $q_x(0,t) = g_1(t)$. Replacing k with $-k$ in the associated global relation we find the following equation, which is valid for $\mathrm{Im}\, k \geq 0$:

$$i\int_0^T e^{ik^2 s} g_1(s)ds = \int_0^\infty e^{ikx} q_0(x)dx - k\int_0^T e^{ik^2 s} g_0(s)ds - e^{ik^2 T}\hat{q}_T(-k). \qquad (18.1)$$

The first two terms on the RHS of (18.1) are known, but the term $\hat{q}_T(-k)$ involves the unknown function $q(x,T)$. Actually, causality implies that $q_x(0,t)$ cannot depend on the "future time" T, and hence the term $\hat{q}_T(-k)$ *cannot* contribute to $q_x(0,t)$. This motivates the following approach for solving (18.1) for $g_1(t)$: The classical Fourier transform inversion formula (after the change of variables $k^2 \to l$) indicates that in order to invert the integral appearing in the LHS of (18.1) we must multiply this integral with $k\exp[-ik^2 t]$. The function $\hat{q}_T(-k)$ is analytic for $\mathrm{Im}\, k > 0$ and the function $\exp[ik^2(T-t)]$ is bounded and analytic for k in the union of the first and third quadrants of the complex k-plane. Hence, the product $k\exp[ik^2(T-t)]\hat{q}_T(-k)$ is bounded and analytic in the first quadrant of the complex k-plane and is of order $O(1)$ as $k \to \infty$. Thus, by integrating around the boundary of the first quadrant, denoted by ∂I, and by appealing to Jordan's lemma (after the change of variables $k^2 \to l$), it follows that the integral of the above product vanishes. Hence, (18.1) yields

$$\pi i g_1(t) = \int_{\partial I} k e^{-ik^2 t}\left[\int_0^\infty e^{ikx} q_0(x)dx - k\int_0^T e^{ik^2 s} g_0(s)ds\right]dk, \quad 0 < t < T. \qquad (18.2)$$

By changing the order of the x- and s-integrations with the k-integration and then by computing the k-integrals (see [43] for details), we find that (18.2) simplifies to the following equation:

$$g_1(t) = -\frac{1}{\sqrt{\pi}} e^{-\frac{i\pi}{4}} \left[\frac{-1}{\sqrt{t}} \int_0^\infty e^{\frac{ix^2}{4t}} \dot{q}_0(x) dx + \int_0^t \frac{\dot{g}_0(s)}{\sqrt{t-s}} ds + \frac{e^{\frac{-i\pi}{4}}}{\sqrt{\pi t}} (q_0(0) - g_0(0)) \right],$$
$$0 < t < T. \tag{18.3}$$

The occurrence of the derivatives $\dot{q}_0$ and $\dot{g}_0$ is due to the fact that in order to obtain well-defined k-integrals we first integrate by parts the x- and s-integrals *before* changing the order of integration.

We now consider the NLS and for brevity of presentation we assume that $q_0(x) = 0$. Then $a(k) = 1$, $b(k) = 0$, and the global relation (16.36) becomes

$$B(k) = e^{4ik^2 T} c^+(k), \quad \operatorname{Im} k \geq 0,$$

where $c^+(k)$ is analytic for $\operatorname{Im} k > 0$ and of order $O(\frac{1}{k})$ as $k \to \infty$. Recalling the definition of $B(k)$ (Definition 16.2) it follows that the global relation can be rewritten in the form

$$\Phi_1(T, k) = -c^+(k), \quad \operatorname{Im} k \geq 0. \tag{18.4}$$

The functions $(\Phi_1(t, k), \Phi_2(t, k))$ are defined by (16.61), and thus we can replace $\Phi_1(T, k)$ by the RHS of (16.61a) evaluated at $t = T$. Thus (18.4) becomes

$$\int_0^T e^{4ik^2\tau} \left\{ i\lambda |g_0(\tau)|^2 \Phi_1(\tau, k) - [2kg_0(\tau) + ig_1(\tau)] \Phi_2(\tau, k) \right\} d\tau$$
$$= -e^{4ik^2 T} c^+(k), \quad \operatorname{Im} k \geq 0. \tag{18.5}$$

In the linear limit, $|g_0|^2 \to 0$ and $\Phi_2 \to 1$, and thus (18.5) reduces to (18.1) (with $q_0 = 0$) after replacing $2k$ with k.

Our goal is to solve (18.5) for $q_x(0, t)$. Comparing (18.1) and (18.5) we see that the main difficulty is the occurrence of the function $\Phi_2(\tau, k)$ in (18.5), which makes the k-dependence prohibitively complicated for the application of the Fourier transform inversion formula. This observation motivates the following question: Does there exist a representation of (Φ_1, Φ_2) involving exponential dependence on k? The answer is positive and such a formula is provided by the so-called Gel'fand–Levitan–Marchenko (GLM) representation. By employing this representation we find that the k-dependence of the integral involving $g_1(\tau)$ is of an explicit exponential type, and then this integral can be solved for $g_1(t)$ using the usual Fourier transform inversion formula.

In what follows we will apply this methodology to the NLS and to the following version of the modified Korteweg–de Vries (mKdV) equation which we will refer to as mKdVI:

$$\text{mKdVI:} \qquad q_t + q_{xxx} - 6\rho q^2 q_x = 0, \quad q \text{ real}, \quad \rho = \pm 1. \tag{18.6}$$

A well-posed problem for mKdVI requires one boundary condition. For concreteness we will analyze the following problems:

$$\text{NLS: given } \; q(x,0)=0, \;\; q(0,t)=g_0(t), \;\; \text{find } \; q_x(0,t)=g_1(t). \tag{18.7}$$

$$\text{mKdVI: given } \; q(x,0)=0, \;\; q(0,t)=g_0(t), \;\; \text{find } \; q_x(0,t)=g_1(t), q_{xx}(0,t)=g_2(t). \tag{18.8}$$

In section 18.1 we will present the GLM representations associated with the t-part of the Lax pair of the NLS and mKdVI equations. In section 18.2 we will employ these representations in order to solve the global relation for the unknown boundary values $g_1(t)$ and $\{g_1(t), g_2(t)\}$, respectively. In section 18.3 we will present an alternative representation of the relevant solutions.

18.1 The Gel′fand–Levitan–Marchenko Representations

The global relation is formulated in terms of the vector $(\Phi_1(t,k), \Phi_2(t,k))$, which is an appropriate solution of the t-part of the associated Lax pair; see Definitions 16.2 and 16.4. The GLM representations of such vectors are given below. For convenience we introduce the functions $\hat{\Phi}_j$, where

$$\hat{\Phi}_j(t,k) = \Phi_j(t,k)e^{if_2(k)t}, \; j=1,2.$$

Proposition 18.1. Let the vector $\hat{\Phi}(t,k) = (\hat{\Phi}_1(t,k), \hat{\Phi}_2(t,k))$ satisfy the following equation:

$$\hat{\Phi}_t + if_2(k)\sigma_3\hat{\Phi} = \tilde{Q}(t,k)\hat{\Phi}, \;\; t>0, \;\; k\in\mathbb{C}, \tag{18.9a}$$

$$\hat{\Phi}(0,k) = (0,1), \tag{18.9b}$$

where the scalar function $f_2(k)$ and the 2×2 matrix-valued function $\tilde{Q}(t,k)$ are as defined below.

NLS

$$f_2 = 2k^2, \quad \tilde{Q}(t,k) = 2kQ_1(t) + Q_0(t),$$

$$Q_1(t) = \begin{pmatrix} 0 & g_0(t) \\ \rho\bar{g}_0(t) & 0 \end{pmatrix}, \quad Q_0(t) = i\rho\begin{pmatrix} -|g_0(t)|^2 & \rho g_1(t) \\ -\bar{g}_1(t) & |g_0(t)|^2 \end{pmatrix}. \tag{18.10}$$

mKdVI

$$f_2 = 4k^3, \quad \tilde{Q}(t,k) = Q_0(t) - 2ikQ_1(t) + 4k^2Q_2(t),$$

$$Q_0(t) = [2\rho g_0^3(t) - g_2(t)]\begin{pmatrix} 0 & 1 \\ \rho & 0 \end{pmatrix}, \quad Q_2(t) = g_0(t)\begin{pmatrix} 0 & 1 \\ \rho & 0 \end{pmatrix}, \tag{18.11}$$

$$Q_1(t) = \rho g_0(t)^2\sigma_3 + g_1(t)\begin{pmatrix} 0 & -1 \\ \rho & 0 \end{pmatrix}.$$

Then the vector $\hat{\Phi}$ can be represented in the following form.

NLS

$$\hat{\Phi}(t,k) = \begin{pmatrix} 0 \\ e^{if_2t} \end{pmatrix} + \int_{-t}^{t}\begin{pmatrix} L_1(t,s) - \frac{i}{2}g_0(t)M_2(t,s) + kM_1(t,s) \\ L_2(t,s) + \frac{i\rho}{2}\bar{g}_0(t)M_1(t,s) + kM_2(t,s) \end{pmatrix} e^{if_2s}ds, \tag{18.12}$$

mKdVI

$$\hat{\Phi}(t,k) = \begin{pmatrix} 0 \\ e^{if_2t} \end{pmatrix} + \int_{-t}^{t}\begin{pmatrix} F_1(t,s,k) \\ F_2(t,s,k) \end{pmatrix} e^{if_2s}ds, \tag{18.13}$$

where

$$F_1 = L_1(t,s) + \frac{1}{2}g_0(t)M_2(t,s) + \frac{1}{4}g_1(t)N_2(t,s) \tag{18.14a}$$
$$+ ik\left(M_1(t,s) - \frac{1}{2}g_0(t)N_2(t,s)\right) + k^2N_1(t,s).$$

$$F_2 = L_2(t,s) - \frac{1}{2}\rho g_0(t)M_1(t,s) + \frac{1}{4}\rho g_1(t)N_1(t,s) \tag{18.14b}$$
$$+ ik\left(M_2(t,s) + \frac{1}{2}\rho g_0(t)N_1(t,s)\right) + k^2N_2(t,s).$$

The functions $\{L_j, M_j, N_j\}_1^2$ satisfy the following equations.

NLS

$$L_1(t,t) = \frac{i}{2}g_1(t), \quad M_1(t,t) = g_0(t), \quad L_2(t,-t) = M_2(t,-t) = 0, \tag{18.15}$$

$$
\begin{aligned}
&L_{1_t} - L_{1_s} = i g_1(t) L_2 + \alpha(t) M_1 + \beta(t) M_2,\\
&L_{2_t} + L_{2_s} = -i\rho \bar{g}_1(t) L_1 - \alpha(t) M_2 + \rho\bar{\beta}(t) M_1,\\
&M_{1_t} - M_{1_s} = 2 g_0(t) L_2 + i g_1(t) M_2,\\
&M_{2_t} + M_{2_s} = 2\rho \bar{g}_0(t) L_1 - i\rho \bar{g}_1(t) M_1,
\end{aligned}
\tag{18.16}
$$

where $\alpha(t)$ and $\beta(t)$ are defined by the equations

$$
\alpha(t) = \frac{\rho}{2}(g_0\bar{g}_1 - \bar{g}_0 g_1), \quad \beta(t) = \frac{1}{2}(i\dot{g}_0 - \rho|g_0|^2 g_0). \tag{18.17}
$$

mKdVI

$$
\begin{gathered}
N_1(t,t) = 2g_0(t), \quad M_1(t,t) = g_1(t), \quad L_1(t,t) = \rho g_0^3(t) - \frac{1}{2} g_2(t),\\
N_2(t,-t) = M_2(t,-t) = L_2(t,-t) = 0,
\end{gathered}
\tag{18.18}
$$

$$
\begin{aligned}
N_{1_t} - N_{1_s} &= [\rho g_0^3(t) - g_2(t)] N_2 - 2 g_1(t) M_2 + 4 g_0(t) L_2,\\
N_{2_t} + N_{2_s} &= [g_0^3(t) - \rho g_2(t)] N_1 + 2\rho g_1(t) M_1 + 4\rho g_0(t) L_1,\\
M_{1_t} - M_{1_s} &= [\rho g_0^3(t) - g_2(t)] M_2 + 2 g_1(t) L_2 + \alpha(t) N_1 - \beta(t) N_2,\\
M_{2_t} + M_{2_s} &= [g_0^3(t) - \rho g_2(t)] M_1 - 2\rho g_1(t) L_1 - \alpha(t) N_2 + \beta(t) N_1,\\
L_{1_t} - L_{1_s} &= [2\rho g_0^3(t) - g_2(t)] L_1 + \gamma(t) M_1 - \frac{1}{2}\dot{g}_0(t) M_2\\
&\quad + \frac{1}{4}\rho g_0(t)\dot{g}_0(t) N_1 + \delta N_2,\\
L_{2_t} + L_{2_s} &= [2 g_0^3(t) - \rho g_2(t)] L_1 - \gamma(t) M_2 + \frac{1}{2}\rho\dot{g}_0(t) M_1\\
&\quad + \frac{1}{4}\rho g_0(t)\dot{g}_0(t) N_2 + \lambda\delta N_1,
\end{aligned}
\tag{18.19}
$$

where α, β, γ, δ are defined by the equations

$$
\begin{gathered}
\alpha(t) = \frac{3}{2} g_0^4 - \rho g_0 g_2 + \frac{1}{2}\rho g_1^2, \quad \beta(t) = \frac{1}{2}(\rho g_0^2 g_1 - \dot{g}_0),\\
\gamma(t) = \rho g_0 g_2 - \frac{1}{2}\rho g_1^2 - \frac{3}{2} g_0^4, \quad \delta(t) = \frac{1}{4}\rho g_0 g_1^2 - \frac{1}{2}\rho g_0^2 g_2 - \frac{1}{4}\dot{g}_1 + \frac{3}{4} g_0^5.
\end{gathered}
\tag{18.20}
$$

Proof. For the derivation of this result we will use the following identity, which can be derived using integration by parts,

$$
\begin{aligned}
&(-i) f_2 \int_{-t}^{t} F(t,s) e^{-i f_2 s \sigma_3} ds\\
&= \left[F(t,t) e^{-i f_2 t \sigma_3} - F(t,-t) e^{i f_2 t \sigma_3} - \int_{-t}^{t} F_s(t,s) e^{-i f_2 s \sigma_3} ds \right] \sigma_3.
\end{aligned}
\tag{18.21}
$$

We will also use

$$\sigma_3 F \sigma_3 = \begin{pmatrix} F_{11} & -F_{12} \\ -F_{21} & F_{22} \end{pmatrix}.$$

The vector $\hat{\Phi}(t,k)$ is the second column vector of the matrix $\hat{\Psi}(t,k)$; the latter matrix satisfies (18.9a) as well as the boundary condition $\hat{\Psi}(0,k) = I$.

NLS

Substituting the representation

$$\hat{\Psi}(t,k) = e^{-if_2 t\sigma_3} + \int_{-t}^{t} [L(t,s) + kM(t,s)] e^{-if_2 s\sigma_3} ds, \tag{18.22}$$

where L and M are 2×2 matrices, into (18.9a) and using (18.21) we find the following equations:

$$M(t,-t) + \sigma_3 M(t,-t)\sigma_3 = 0, \tag{18.23a}$$

$$L(t,-t) + \sigma_3 L(t,-t)\sigma_3 + iQ_1(t)M(t,-t)\sigma_3 = 0, \tag{18.23b}$$

$$M(t,t) - \sigma_3 M(t,t)\sigma_3 = 2Q_1(t), \tag{18.24a}$$

$$L(t,t) - \sigma_3 L(t,t)\sigma_3 - iQ_1(t)M(t,t)\sigma_3 = Q_0(t), \tag{18.24b}$$

$$M_t(t,s) + \sigma_3 M_s(t,s)\sigma_3 = 2Q_1(t)L(t,s) + Q_0(t)M(t,s), \tag{18.25a}$$

$$L_t(t,s) + \sigma_3 L_s(t,s)\sigma_3 = -iQ_1(t)M_s(t,s)\sigma_3 + Q_0(t)L(t,s). \tag{18.25b}$$

Equations (18.23b) and (18.24b) suggest replacing the function L by $\tilde{L}$, where

$$L(t,s) = \tilde{L}(t,s) + \frac{i}{2} Q_1(t)\sigma_3 M(t,s). \tag{18.26}$$

Using this equation in (18.23)–(18.25) we find the following equations:

$$M(t,-t) + \sigma_3 M(t,-t)\sigma_3 = 0, \tag{18.27a}$$

$$\tilde{L}(t,-t) + \sigma_3 \tilde{L}(t,-t)\sigma_3 = 0, \tag{18.27b}$$

$$M(t,t) - \sigma_3 M(t,t)\sigma_3 = 2Q_1(t), \tag{18.28a}$$

$$\tilde{L}(t,t) - \sigma_3 \tilde{L}(t,t)\sigma_3 = Q_0(t), \tag{18.28b}$$

$$M_t + \sigma_3 M_s \sigma_3 = 2Q_1(t)\tilde{L} + [iQ_1^2(t)\sigma_3 + Q_0(t)]M, \tag{18.29a}$$

$$\begin{aligned} \tilde{L}_t + \sigma_3 \tilde{L}_s \sigma_3 = {} & [Q_0(t) - iQ_1(t)\sigma_3 Q_1(t)]\tilde{L} \\ & + \frac{i}{2}[Q_0(t)Q_1(t)\sigma_3 - iQ_1^3(t) - Q_1(t)\sigma_3 Q_0(t) - \dot{Q}_1(t)\sigma_3]M. \end{aligned} \tag{18.29b}$$

For the derivation of (18.27b) we have used the identity

$$Q_1(t)\sigma_3 M(t,-t) - Q_1(t)M(t,-t)\sigma_3 + 2Q_1(t)M(t,-t)\sigma_3 = 0, \tag{18.30}$$

which follows from the fact that $M(t,-t)$ is off-diagonal (see (18.23a)). Similarly, for the derivation of (18.28b) we have used an equation similar to (18.30), where $M(t,-t)$ is replaced by the matrix $M(t,t)$ which is also off-diagonal.

Equations (18.27)–(18.29) imply that the 2×2 matrices $\tilde{L}$ and M have the following structure:

$$\tilde{L} = \begin{pmatrix} \bar{L}_2 & L_1 \\ \rho\bar{L}_1 & L_2 \end{pmatrix}, \quad M = \begin{pmatrix} \bar{M}_2 & M_1 \\ \rho\bar{M}_1 & M_2 \end{pmatrix}. \tag{18.31}$$

Using these representations, (18.27)–(18.29) imply (18.16). Furthermore, the second column vector of (18.22) yields (18.12).

mKdVI

Substituting the representation

$$\hat{\Psi}(t,k) = e^{-if_2 t\sigma_3} + \int_{-t}^{t} [L(t,s) + ikM(t,s) + k^2 N(t,s)] e^{-if_2 s\sigma_3} ds \tag{18.32}$$

into the first of equations (18.9a) and using (18.21) we find the following equations:

$$N(t,-t) + \sigma_3 N(t,-t)\sigma_3 = 0, \tag{18.33a}$$

$$M(t,-t) + \sigma_3 M(t,-t)\sigma_3 + Q_2(t)N(t,-t)\sigma_3 = 0, \tag{18.33b}$$

$$L(t,-t) + \sigma_3 L(t,-t)\sigma_3 + \frac{1}{2}Q_1(t)N(t,-t)\sigma_3 - Q_2(t)M(t,-t)\sigma_3 = 0, \tag{18.33c}$$

$$N(t,t) - \sigma_3 N(t,t)\sigma_3 = 4Q_2(t), \tag{18.34a}$$

$$M(t,t) - \sigma_3 M(t,t)\sigma_3 = -2Q_1(t) + Q_2(t)N(t,t)\sigma_3, \tag{18.34b}$$

$$L(t,t) - \sigma_3 L(t,t)\sigma_3 = Q_0(t) + \frac{1}{2}Q_1(t)N(t,t)\sigma_3 - Q_2(t)M(t,t)\sigma_3, \tag{18.34c}$$

$$N_t + \sigma_3 N_s \sigma_3 = Q_0(t)N + 2Q_1(t)M + 4Q_2(t)L, \tag{18.35a}$$

$$M_t + \sigma_3 M_s \sigma_3 = Q_0(t)M - 2Q_1(t)L - Q_2(t)N_s\sigma_3, \tag{18.35b}$$

$$L_t + \sigma_3 L_s \sigma_3 = Q_0(t)L - \frac{1}{2}Q_1(t)N_s\sigma_3 + Q_2(t)M_s\sigma_3. \tag{18.35c}$$

Equations (18.33c), (18.34b), and (18.34c) suggest replacing M and L by $\tilde{M}$ and $\tilde{L}$, where

$$M = \tilde{M} + \frac{1}{2}Q_2(t)\sigma_3 N,$$

$$L = \tilde{L} - \frac{1}{2}Q_2(t)\sigma_3\tilde{M} + \frac{1}{4}g_1(t)\Lambda, \tag{18.36}$$

and Λ denotes

$$\Lambda = \begin{pmatrix} 0 & 1 \\ \rho & 0 \end{pmatrix}. \tag{18.37}$$

Using (18.36), (18.33)–(18.35) become

$$
\begin{aligned}
&F(t,-t)+\sigma_3 F(t,-t)\sigma_3 = 0, \quad F = N \text{ or } \tilde{M} \text{ or } \tilde{L},\\
&N(t,t)-\sigma_3 N(t,t)\sigma_3 = 4g_0(t)\Lambda,\\
&\tilde{M}(t,t)-\sigma_3\tilde{M}(t,t)\sigma_3 = -2g_1(t)\Lambda\sigma_3,\\
&\tilde{L}(t,t)-\sigma_3\tilde{L}(t,t)\sigma_3 = [2\rho g_0^3(t)-g_2(t)]\Lambda,
\end{aligned}
$$

$$
\begin{aligned}
N_t+\sigma_3 N_s\sigma_3 &= [\rho g_0^3(t)-g_2(t)]\Lambda N + 2g_1(t)\Lambda\sigma_3\tilde{M} + 4g_0(t)\Lambda\tilde{L},\\
\tilde{M}_t+\sigma_3\tilde{M}_s\sigma_3 &= [\rho g_0^3(t)-g_2(t)]\Lambda\tilde{M} + [\alpha(t)+\beta(t)\Lambda]\sigma_3 N - 2g_1\Lambda\sigma_3\tilde{L},\\
\tilde{L}_t+\sigma_3\tilde{L}_s\sigma_3 &= [2\rho g_0^3(t)-g_2(t)]\Lambda\tilde{L} + \left[\gamma(t)+\frac{1}{2}\dot{g}_0(t)\Lambda\right]\sigma_3\tilde{M}\\
&\quad + \left[\delta(t)\Lambda+\frac{1}{4}\rho g_0(t)\dot{g}_0(t)\right]N,
\end{aligned}
\tag{18.38}
$$

where $\alpha, \beta, \gamma, \delta$ are as defined by equations (18.20).

These equations imply that $N, \tilde{M}, \tilde{L}$ have the following structure:

$$
N=\begin{pmatrix} N_2 & N_1\\ \rho N_1 & N_2\end{pmatrix},\quad \tilde{M}=\begin{pmatrix} -M_2 & M_1\\ -\rho M_1 & M_2\end{pmatrix},\quad \tilde{L}=\begin{pmatrix} L_2 & L_1\\ \rho L_1 & L_2\end{pmatrix}. \tag{18.39}
$$

Using these representations, equations (18.38) imply equations (18.19), and the second column vector of (18.32) yields (18.13). ∎

Remark 18.1. In the subsequent analysis it is more convenient to use integrals from 0 to t instead of integrals from $-t$ to t. In this respect, using the identity

$$
\int_{-t}^{t} F(t,s,k)e^{if_2 s}ds = 2e^{-if_2 t}\int_0^t F(t,2\tau-t,k)e^{2if_2\tau}d\tau,
$$

we find that the GLM representations of the NLS and mKdVI take the following form:

NLS

$$
\hat{\Phi}_1(t,k) = 2e^{-if_2 t}\int_0^t \left(L_1-\frac{i}{2}g_0(t)M_2+kM_1\right)e^{2if_2\tau}d\tau, \tag{18.40a}
$$

$$
\hat{\Phi}_2(t,k) = e^{if_2 t}+2e^{-if_2 t}\int_0^t \left(L_2+\frac{i\rho}{2}\bar{g}_0(t)M_1+kM_2\right)e^{2if_2\tau}d\tau, \tag{18.40b}
$$

where L_1, L_2, M_1, M_2 are functions of $(t, 2\tau-t)$.

mKdVI

$$\hat{\Phi}_1(t,k) = 2e^{-if_2t}\int_0^t F_1(t, 2\tau - t, k)e^{2if_2\tau}d\tau, \tag{18.41a}$$

$$\hat{\Phi}_2(t,k) = e^{if_2t} + 2e^{-if_2t}\int_0^t F_2(t, 2\tau - t, k)e^{2if_2\tau}d\tau, \tag{18.41b}$$

where F_1, F_2 are as defined by (18.14).

18.2 The Solution of the Global Relation in Terms of the GLM Functions

In what follows we will analyze the global relation (18.4) together with the GLM representation of $\hat{\Phi}_1$ in order to find explicit expressions for the unknown boundary values. For this purpose it is more convenient to use the global relation corresponding to t instead of T.

Proposition 18.2. Suppose that the function $\hat{\Phi}_1(t,k)$ satisfies the global relation

$$\hat{\Phi}_1(t,k) = -e^{if_2(k)t}C(k,t), \quad \operatorname{Im} k \geq 0, \quad 0 < t < T, \tag{18.42}$$

where $C(k,t)$ is an analytic function of k for $\operatorname{Im} k > 0$ and of order $O(\frac{1}{k})$ as $k \to \infty$ for all $0 < t < T$.

(a) NLS

Let $\hat{\Phi}_1$ be given in terms of the functions $L_1(t,\tau)$, $M_1(t,\tau)$, and $M_2(t,\tau)$ by (18.40a), where

$$L_1(t,t) = \frac{i}{2}g_1(t), \quad M_1(t,-t) = 0. \tag{18.43}$$

Then $g_1(t)$ (the Neumann boundary value) can be expressed in terms of $g_0(t)$ (the Dirichlet boundary value) and of the functions M_1, M_2 by the equation

$$g_1(t) = g_0(t)M_2(t,t) - \frac{e^{-\frac{i\pi}{4}}}{\sqrt{\pi}}\int_0^t \frac{\partial M_1}{\partial \tau}(t, 2\tau - t)\frac{d\tau}{\sqrt{t-\tau}}. \tag{18.44}$$

(b) mKdVI

Let $\hat{\Phi}_1$ be given in terms of the functions

$$L_1(t,\tau), \quad \left\{M_j(t,\tau), \; N_j(t,\tau)\right\}_{j=1}^2$$

by (18.41a), where

$$M_1(t,t) = g_1(t), \quad L_1(t,t) = -\frac{1}{2}g_2(t) + \rho g_0^3(t). \tag{18.45}$$

Then, $g_1(t)$ (the Neumann boundary value) and $g_2(t) = q_{xx}(0,t)$ can be expressed in terms of $g_0(t)$ (the Dirichlet boundary value) and of the functions N_1, N_2, M_2 by the following equations:

$$g_1(t) = \frac{1}{2}g_0(t)N_2(t,t) + \frac{3c}{2\pi}\left[\frac{N_1(t,-t)}{t^{\frac{1}{3}}} + \int_0^t \frac{\partial N_1}{\partial \tau}(t, 2\tau - t)\frac{d\tau}{(t-\tau)^{\frac{1}{3}}}\right] \tag{18.46}$$

and

$$\begin{aligned} g_2(t) &= 2\rho g_0^3(t) + g_0(t)M_2(t,t) + \frac{1}{2}g_1(t)N_2(t,t) \\ &+ \frac{3i\tilde{c}}{\pi}\left[\frac{N_1(t,-t)}{t^{\frac{2}{3}}} + \int_0^t \frac{\partial N_1}{\partial \tau}(t, 2\tau - t)\frac{d\tau}{(t-\tau)^{\frac{2}{3}}}\right], \end{aligned} \tag{18.47}$$

where

$$c = -\frac{\Gamma(\frac{1}{3})}{2\sqrt{3}}, \quad \tilde{c} = \frac{i\Gamma(\frac{2}{3})}{4\sqrt{3}}.$$

Proof. (a) Replacing the function $\hat{\Phi}_1(t,k)$ by the RHS of equation (18.40a) in the global relation (18.42) we find

$$\begin{aligned} &\int_0^t e^{4ik^2\tau}\left[L_1(t, 2\tau - t) - \frac{i}{2}g_0(t)M_2(t, 2\tau - t) + kM_1(t, 2\tau - t)\right]d\tau \\ &\quad = -\frac{e^{4ik^2t}}{2}C(k,t), \quad \operatorname{Im} k \geq 0, \quad 0 < t < T. \end{aligned} \tag{18.48}$$

We multiply this equation by $k\exp[-4ik^2t']$, $t' < t$, and integrate along the boundary of the first quadrant of the complex k-plane, which we denote by ∂I, with the orientation shown in Figure 18.1.

The RHS of the resulting equation vanishes because $kC(k,t)$ is analytic and of order $O(1)$ for $\operatorname{Im} k > 0$, and the oscillatory term $\exp[ik^2(t-t')]$ is bounded in the first quadrant.

The first two terms on the LHS of (18.48) give contributions which can be computed in closed form using the following identity:

$$\int_{\partial I} k\left[\int_0^t e^{4ik^2(\tau - t')}K(\tau,t)d\tau\right]dk = \frac{\pi}{4}K(t',t), \quad t > 0, \ t' > 0, \ t' < t, \tag{18.49}$$

where $K(\tau,t)$ is a smooth function of its arguments; if $t = t'$ the 1/4 is replaced by 1/8. This identity follows from the usual Fourier transform identity after using the transformation $4k^2 \to l$ to map ∂I to the real axis. Using this identity, the first two terms on the LHS of (18.48) yield

$$\frac{\pi}{4}\left[L_1(t, 2t' - t) - \frac{i}{2}g_0(t)M_2(t, 2t' - t)\right]. \tag{18.50}$$

Before computing the contribution of the third term in the LHS of (18.48), we first use integration by parts:

$$\begin{aligned} \int_0^t ke^{4ik^2\tau}M_1(t, 2\tau - t)d\tau &= \frac{1}{4ik}\left[e^{4ik^2t}M_1(t,t) - M_1(t,-t)\right] \\ &\quad - \frac{1}{4ik}\int_0^t e^{4ik^2\tau}\frac{\partial M_1}{\partial \tau}(t, 2\tau - t)d\tau. \end{aligned} \tag{18.51}$$

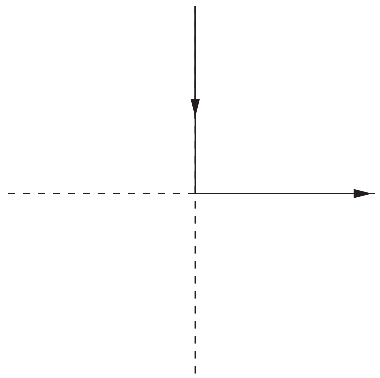


Figure 18.1. *The curve* ∂I.

Multiplying the third term of the LHS of (18.48) by $k\exp[-4ik^2t']$ and integrating along ∂I, we find three contributions. The first vanishes due to the fact that $\exp[4ik^2(t-t')]$ is bounded in the first quadrant of the complex k-plane. The k-integral of the second contribution can be computed in closed form as follows: Using $k^2t' = l^2$ we find

$$\int_{\partial I} e^{-4ik^2t'}dk = \int_{\partial I} e^{-4il^2}\frac{dl}{\sqrt{t'}} = \frac{c}{\sqrt{t'}}$$

with

$$c = \int_{\partial I} e^{-4il^2}dl = \frac{1}{2}\int_{\partial I} e^{-il^2}dl = \frac{1}{2}\int_{-\infty}^{\infty} e^{-il^2}dl = \int_0^{\infty} e^{-il^2}dl = \frac{1}{2}e^{-\frac{i\pi}{4}}\Gamma\left(\frac{1}{2}\right), \tag{18.52}$$

where in deforming ∂I to the real axis we have used the fact that $\exp(-il^2)$ is bounded in the second quadrant of the complex k-plane.

In order to compute the contribution of the third term in the RHS of (18.51) we split $\int_0^t$ into $\int_0^{t'}$ and $\int_{t'}^t$. The contribution of the second integral vanishes due to analyticity considerations, and the contribution of the first integral yields a k-integral which equals $c/\sqrt{t'-t}$. Thus, (18.48) yields

$$\frac{\pi}{4}\left[L_1(t, 2t'-t) - \frac{i}{2}g_0(t)M_2(t, 2t'-t)\right] - \frac{c}{4i}\left[\frac{M_1(t,-t)}{\sqrt{t'}} + \int_0^{t'}\frac{\partial M_1}{\partial \tau}(t, 2\tau - t)\frac{d\tau}{\sqrt{t'-\tau}}\right] = 0.$$

Letting $t' \to t$ and using (18.43), the above equation becomes (18.44).

(b) Replacing $\hat{\Phi}_1(t,k)$ by the RHS of equation (18.41a) in the global relation (18.42) we find

$$2\int_0^t F_1(t, 2\tau - t, k)e^{8ik^3\tau}d\tau = -e^{8ik^3t}C(k,t), \quad 0 < t < T, \quad \operatorname{Im} k \geq 0, \tag{18.53}$$

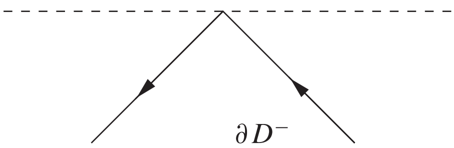


Figure 18.2. *The contour ∂D^-.*

where F_1 is defined by (18.14a). We replace k by αk and by $\alpha^2 k$, in (18.53) where $\alpha = \exp[2i\pi/3]$. Then we obtain two equations valid in D^-, where

$$D^- = \left\{k \in \mathbb{C},\ \frac{4\pi}{3} < \arg k < \frac{5\pi}{3}\right\}. \tag{18.54}$$

Solving these two equations for the two integrals containing

$$ik\left(M_1 - \frac{1}{2}g_0(t)N_2\right), \quad L_1 + \frac{1}{2}g_0(t)M_2 + \frac{1}{4}g_1(t)N_2,$$

we find the following two equations which are valid in D^-:

$$2ik\int_0^t e^{8ik^2\tau}\left[M_1 - \frac{1}{2}g_0(t)N_2\right]d\tau = 2k^2\int_0^t e^{8ik^3\tau}N_1 d\tau$$
$$+\frac{e^{8ik^3t}}{\alpha^2-\alpha}\left[C(\alpha k,t) - C(\alpha^2 k,t)\right], \tag{18.55}$$

$$2\int_0^t e^{8ik^3\tau}\left[L_1 + \frac{1}{2}g_0(t)M_2 + \frac{1}{4}g_1(t)N_2\right]d\tau = 2k^2\int_0^t e^{8ik^3\tau}N_1 d\tau$$
$$+\frac{e^{8ik^3t}}{\alpha-1}\left[C(\alpha^2 k,t) - \alpha C(\alpha k,t)\right]. \tag{18.56}$$

We first analyze (18.55). We multiply this equation by $k\exp[-8ik^3t']$, $t' < t$, and integrate along the boundary of D^- with the orientation shown in Figure 18.2.
The integral appearing in the LHS of (18.55) can be inverted using the following identity:

$$\int_{\partial D^-} k^2\left[\int_0^t e^{8ik^3(\tau-t')}K(\tau,t)d\tau\right]dk = \frac{\pi}{12}K(t',t), \quad t>0,\ t'>0,\ t'<t, \tag{18.57}$$

where $K(\tau,t)$ is a smooth function of its arguments; if $t = t'$ the 1/12 is replaced by 1/24 (this identity follows from the usual Fourier transform identity after using the transformation $8i^3 \to l$ to map ∂D^- to the real axis). Using this identity and recalling that M_1 and N_2 are evaluated at $(t, 2\tau - t)$, the LHS of (18.55) yields

$$2i\frac{\pi}{12}\left[M_1(t,2t'-t) - \frac{1}{2}g_0(t)N_2(t,2t'-t)\right].$$

The second term on the RHS of (18.55) yields a term which is bounded and analytic in D^- and of order $O(1)$ as $k \to \infty$, and thus it yields a zero contribution (using Jordan's lemma and the substitution $8k^3 \to l$). Before computing the contribution of the integral appearing in the RHS of (18.55), we first integrate by parts as follows:

$$2k^3 \int_0^t e^{8ik^3\tau} N_1(t, 2\tau - t) d\tau = \frac{1}{4i}\left[e^{8ik^3 t} N_1(t,t) - N_1(t,-t)\right]$$
$$-\frac{1}{4i}\int_0^t e^{8ik^3\tau}\frac{\partial}{\partial\tau}N_1(t, 2\tau - t)d\tau. \tag{18.58}$$

Multiplying by $\exp[-8ik^3t']$, $t' < t$, and integrating along ∂D^- we find three contributions. Proceeding as in (a), we find that the contribution associated with $N_1(t,t)$ vanishes and that the contribution associated with $N_1(t,-t)$ yields (after the change of variables $k^3t' = l^3$)

$$\int_{\partial D^-} e^{-8ik^3t'}dk = \int_{\partial D^-} e^{-8il^3}\frac{dl}{t'^{\frac{1}{3}}} = \frac{c}{(t')^{\frac{1}{3}}}, \quad c = \int_{\partial D^-} e^{-8il^3}dl,$$

whereas the contribution associated with $\frac{\partial N_1}{\partial\tau}$ yields

$$\int_0^{t'} e^{-8ik^3(t'-\tau)}dk = \frac{c}{(t'-\tau)^{\frac{1}{3}}}.$$

Hence, (18.55) yields

$$\frac{i\pi}{6}\left[M_1(t, 2t'-t) - \frac{1}{2}g_0(t)N_2(t, 2t'-t)\right]$$
$$= -\frac{c}{4i}\left[\frac{N_1(t,-t)}{(t')^{\frac{1}{3}}} + \int_0^{t'}\frac{\partial N_1}{\partial\tau}(t, 2\tau - t)\frac{d\tau}{(t'-\tau)^{\frac{1}{3}}}\right].$$

Taking the limit of this equation as $t' \to t$ and using (18.45) we find (18.46), where for the computation of the constant c we use the contour deformation to find (see [17])

$$c = \int_{\partial D^-} e^{-8il^3}dl = \frac{1}{2}\int_{\partial D^-} e^{-il^3}dl = -\frac{\Gamma\left(\frac{1}{3}\right)}{2\sqrt{3}}.$$

We now analyze (18.56). We multiply this equation by $k^2\exp[-8ik^3t']$, $t' < t$, and proceed as above. The LHS of the resulting equation can be inverted explicitly using (18.57), and the second term on the RHS of (18.56) yields a zero contribution. The first term on the RHS of (18.56) yields a term which equals the RHS of (18.58) times $k\exp[-8ik^3t']$. This term gives rise to terms involving powers of 2/3 instead of 1/3, as well as $\tilde{c}$ instead of c, where

$$\tilde{c} = \int_{\partial D^-} le^{-8il^3}dl.$$

For example, the term associated with $N_1(t,-t)$ yields

$$\int_{\partial D^-} ke^{-8ik^3t'}dk = \int_{\partial D^-} le^{-8il^3}\frac{dl}{(t')^{\frac{2}{3}}} = \frac{\tilde{c}}{(t')^{\frac{2}{3}}}.$$

Hence, (18.56) yields

$$\frac{\pi}{6}\left[L_1(t,2t'-t)+\frac{1}{2}g_0(t)M_2(t,2t'-t)+\frac{1}{4}g_1(t)N_2(t,2t'-t)\right]$$

$$=-\frac{\tilde{c}}{4i}\left[\frac{N_1(t,-t)}{(t')^{\frac{2}{3}}}+\int_0^{t'}\frac{\partial N_1}{\partial\tau}(t,2\tau-t)\frac{d\tau}{(t'-\tau)^{\frac{1}{3}}}\right].$$

Taking the limit of this equation as $t'\to t$ and using equations (18.45) we find (18.47), where the constant $\tilde{c}$ is given by

$$\tilde{c}=\int_{\partial D^-} le^{-8il^3}dl=\frac{1}{4}\int_{\partial D^-} le^{-il^3}dl=\frac{i\Gamma\left(\frac{2}{3}\right)}{4\sqrt{3}}.$$ □

Remark 18.2. In the linear limit of the NLS equation, the definitions of $\{L_j, M_j\}_1^2$ imply

$$L_1(t,s)\to L_1(t+s),\quad M_1(t,s)\to M_1(t+s),\quad L_2\to 0,\quad M_2\to 0.$$

Thus using the conditions at $s=t$, i.e., (18.15), we find

$$L_1(t,t)\to L_1(2t)=\frac{i}{2}g_1(t),\ \ M_1(t,t)\to M_1(2t)=g_0(t),$$

$$M_1(t,2\tau-t)\to M_1(2\tau)=g_0(\tau).$$

Hence, the linear limit of (18.44) yields

$$g_1(t)=-\frac{e^{-\frac{i\pi}{4}}}{\sqrt{\pi}}\int_0^t\frac{\partial g_0(\tau)}{\partial\tau}\frac{d\tau}{\sqrt{t-\tau}},$$

which coincides with the Dirichlet to Neumann map of (72), with $q(x,0)=0$; see (18.3).

18.3 The Solution of the Global Relation in Terms of $\Phi(t,k)$

Proposition 18.2 presents the solution of the global relation in terms of the functions $\{L_j, M_j, N_j\}_1^2$ appearing in the associated GLM representations. In what follows we will express the solution directly in terms of $\Phi(t,k)$.

Proposition 18.3. Suppose that the function $\Phi_1(t,k)$ satisfies the global relation (18.4).

(a) NLS

Let $\Phi_1(t,k)$, $\Phi_2(t,k))$ be given in terms of the functions $g_0(t)$ and $g_1(t)$ by (16.61) with $\tilde{Q}(t,k)$ defined by (16.57). Then $g_1(t)$ (the Neumann boundary value) can be expressed in terms of $g_0(t)$ (the Dirichlet boundary value) and of the functions $(\Phi_1(t,k), \Phi_2(t,k))$ by the equation

$$g_1(t) = \frac{2g_0(t)}{\pi}\int_{\partial I}[\Phi_2(t,k) - \Phi_2(t,-k)]\,dk$$
$$+\frac{2i}{\pi}\int_{\partial I}(k[\Phi_1(t,k) - \Phi_1(t,-k)] + ig_0(t))\,dk, \tag{18.59}$$

where ∂I denotes the boundary of the first quadrant of the complex k-plane with the orientation shown in Figure 18.1.

(b) mKdVI

Let $(\Phi_1(t,k), \Phi_2(t,k))$ be given in terms of the functions $\{g_j(t)\}_0^2$ by (16.133) with $f_2(t)$ and $\tilde{Q}(t,k)$ as defined by (18.11). Then $g_1(t)$ (the Neumann boundary values) and $g_2(t) = q_{xx}(0,t)$ can be expressed in terms of $g_0(t)$ (the Dirichlet boundary value) and of the functions $(\Phi_1(t,k), \Phi_2(t,k))$ by the following equations:

$$g_1(t) = \frac{2g_0}{\pi}\int_{\partial D^-}\chi_2(t,k)dk + \frac{6}{i\pi}\int_{\partial D_-}\left[\frac{k}{3}\chi_1(t,k) - \frac{g_0(t)}{2i}\right]dk \tag{18.60}$$

and

$$g_2(t) = \rho g_0^3(t) - \frac{4ig_0(t)}{\pi}\int_{\partial D^-}k\hat{\chi}_2(t,k)dk + \frac{2g_1(t)}{\pi}\int_{\partial D^-}\chi_2(t,k)dk$$
$$-\frac{12}{\pi}\int_{\partial D^-}\left[\frac{1}{3}k^2\chi_1(t,k) - \frac{k}{2i}g_0(t)\right]dk, \tag{18.61}$$

where χ_1, χ_2, and $\hat{\chi}$ are defined by

$$\chi_1(t,k) = \Phi_1(t,k) + \alpha\Phi_1(t,\alpha k) + \alpha^2\Phi_1(t,\alpha^2 k),$$
$$\chi_2(t,k) = \Phi_2(t,k) + \alpha\Phi_2(t,\alpha k) + \alpha^2\Phi_2(t,\alpha^2 k), \quad \alpha = e^{\frac{2i\pi}{3}},$$
$$\widehat{\chi}_2(t,k) = \Phi_2(t,k) + \alpha^2\Phi_2(t,\alpha k) + \alpha\Phi_2(t,\alpha^2 k), \tag{18.62}$$

and ∂D^- is the boundary of the domain D^- defined in (18.54) with the orientation shown in Figure 18.2.

Proof. (a) In order to derive (18.59), in addition to (18.49), we will also need the following identity:

$$\int_{\partial I}k^2\left[\int_0^t e^{4ik^2(\tau-t')}K(\tau,t)d\tau\right]dk$$
$$= \int_{\partial I}k^2\left[\int_0^{t'} e^{4ik^2(\tau-t')}K(\tau,t)d\tau - \frac{K(t',t)}{4ik^2}\right]dk, \quad t>0, t'>0, t'<t, \tag{18.63}$$

where ∂I is the oriented boundary of the first quadrant of the complex k-plane and $K(\tau, t)$ is a smooth function of the arguments indicated. This identity can be derived as follows:

We rewrite the LHS of (18.63) as the RHS plus the term

$$\int_{\partial I} k^2 \left[\int_{t'}^{t} e^{4ik^2(\tau - t')} K(\tau, t) d\tau + \frac{K(t', t)}{4ik^2} \right] dk.$$

The integrand of the above integral is analytic and bounded in the domain enclosed by ∂I. Also, its zero order term with respect to $(k^2)^{-1}$ is given by $K(t,t)\exp[4ik^2(t - t')]/4i$; thus Jordan's lemma applied in the first quadrant of the complex k-plane implies the desired result.

The main difference between the derivation of (18.59) and (18.44) is that in what follows we will *not* compute some of the relevant k-integrals. In this respect we will first derive the following equation:

$$\begin{aligned} -\frac{\pi}{8i} g_1(t) = &-\frac{i}{2} g_0(t) \int_{\partial I} k \left[\int_0^t e^{4ik^2(\tau - t)} M_2(t, 2\tau - t) d\tau \right] dk \\ &+ \frac{1}{2} \int_{\partial I} \left[2k^2 \int_0^t e^{4ik^2(\tau - t)} M_1(t, 2\tau - t) d\tau - \frac{g_0(t)}{2i} \right] dk. \end{aligned} \tag{18.64}$$

Indeed, multiplying the global relation (18.48) by $k\exp[-4k^2 t']$, $t' < t$, and integrating along ∂I we find

$$\begin{aligned} \int_{\partial I} \Bigg[\int_0^t e^{4ik^2(\tau - t')} \Big(& kL_1(t, 2\tau - t) - \tfrac{i}{2} g_0(t) k M_2(t, 2\tau - t) \\ & + k^2 M_1(t, 2\tau - t) d\tau \Big) \Bigg] dk = 0. \end{aligned} \tag{18.65}$$

We recall that $L_1(t,t) = ig_1(t)/2$ and $M_1(t,t) = g_0(t)$, but $M_2(t,t)$ is unknown. Hence, we will manipulate the first and third terms of the LHS of (18.65) but not the second term. Using (18.49) and (18.63) to simplify the former two terms, (18.65) yields

$$\begin{aligned} &\frac{\pi}{4} L_1(t, 2t' - t) - \frac{i}{2} g_0(t) \int_{\partial I} k \left[\int_0^t e^{4ik^2(\tau - t')} M_2(t, 2\tau - t) d\tau \right] dk \\ &\int_{\partial I} \left[k^2 \int_0^{t'} e^{4ik^2(\tau - t')} M_1(t, 2\tau - t) d\tau - \frac{M_1(t, 2t' - t)}{4i} \right] dk = 0. \end{aligned} \tag{18.66}$$

The second term in the LHS of this equation equals

$$-ig_0(t)\pi \frac{M_2(t, 2t' - t)}{8},$$

and thus the integral is well defined.

Letting $t' \to t$ in (18.66) and using (18.43), (18.15) and (18.49), we find (18.64).

This equation expresses $g_1(t)$ in terms of $g_0(t)$ and of integrals involving M_2 and M_1. These latter integrals can be expressed in terms of $\hat{\Phi}_1$ and $\hat{\Phi}_2$. Indeed, replacing k by $-k$ in (18.40a) and then subtracting the resulting equation from (18.40a), we find

$$4ke^{-2ik^2t}\int_0^t e^{4ik^2\tau}M_1(t,2\tau-t)d\tau = \hat{\Phi}_1(t,k)-\hat{\Phi}_1(t,-k). \quad (18.67a)$$

Similarly, (18.40b) and the equation obtained from (18.40b) by replacing k with $-k$ yield

$$4ke^{-2ik^2t}\int_0^t e^{4ik^2\tau}M_2(t,2\tau-t)d\tau = \hat{\Phi}_2(t,k)-\hat{\Phi}_2(t,-k). \quad (18.67b)$$

Using (18.67) in (18.64), the latter equation becomes (18.59), where we have used that $\hat{\Phi}_j = \Phi_j e^{2ik^2t}$, $j = 1, 2$.

(b) The derivation is conceptually similar with that of (a) details can be found in [58]. □

Remark 18.3. In the case of the NLS, (18.44) expresses g_1 in terms of $\{g_0, M_2, M_1\}$; replacing g_1 by this latter expression in (18.15)–(18.17) we find a *nonlinear* hyperbolic system for $\{L_j, M_j\}_1^2$. Similarly, (18.59) expresses g_1 in terms of $\{g_0, \Phi_1, \Phi_2\}$; replacing g_1 by this latter expression in equations (16.61) we find a system of *nonlinear* equations for $\{\Phi_1, \Phi_2\}$. This system involves only *two* equations, whereas the system for $\{L_j, M_j\}_1^2$ involves *four* equations. Similarly, for the mKdV equation, the nonlinear system for $\{\Phi_1, \Phi_2\}$ still involves only *two* equations, whereas the nonlinear system for $\{L_j, M_j, N_j\}_1^2$ involves *six* equations. Thus, it appears that the formulation presented in Proposition 18.3 has an analytical advantage. On the other hand, it was shown in [123] that the expressions in Proposition 18.2 are more convenient for the numerical evaluation of the Dirichlet to Neumann correspondence. The rigorous analysis of the above nonlinear systems remains open.

Remark 18.4. The first explicit solution of the global relation in the case of the NLS was obtained in [57], where $g_1(t)$ was expressed in terms of $g_0(t)$ and of the functions $\{\lambda_j(t,k^2), \mu_j(t,k^2)\}_1^2$ which were defined in terms of $\{L_j(t,s), M_j(t,s)\}_1^2$. It was later realized in [58] that the functions $\{\lambda_j, \mu_j\}_1^2$ can be expressed in terms of $\{\Phi_1, \Phi_2\}$, and thus the Dirichlet to Neumann map for the NLS is characterized in [58] in terms of *two* ODEs; see Remark 18.3. The sG as well as the two versions of mKdV (namely, mKdVI and mKdVII) are analyzed in [58], without the assumption of $q(x,0) = 0$. The two versions of the KdV equation are analyzed in [59].

Remark 18.5. The GLM representation for $\hat{\Phi}(t,k)$ was first derived in [111]. The analogous representations for the sG, mKdVI, and mKdVII are presented in [58]. The construction of the GLM representations for the two versions of KdV presents some novel difficulties which were finally overcome in [59].

Chapter 19

Asymptotics of Oscillatory Riemann–Hilbert Problems

It was shown in Chapter 16 that the solution of nonlinear integrable dispersive evolution PDEs formulated on the half-line can be expressed in terms of the solution of oscillatory matrix-valued Riemann–Hilbert (RH) problems. Similar results for nonlinear integrable PDEs formulated on the finite interval are presented in [54], [55]. The main advantage of this formalism is that it provides an effective approach for studying asymptotic properties of the solution. For example, by employing Theorem 16.1 and by using the Deift–Zhou–Venakides asymptotic analysis of oscillatory RH problems, Kamvissis [71] computed the semiclassical limit of the NLS formulated on the half-line.

In section 19.1 we will compute the large-t limit of the solution $q(x,t)$ of the NLS characterized through Theorem 16.1. In section 19.2 we will compute a certain asymptotic limit of three coupled nonlinear PDEs describing the phenomenon of stimulated Raman scattering.

19.1 The Large-*t* Limit of the Nonlinear Schrödinger Equation on the Half-Line

Theorem 16.1 expresses the solution $q(x,t)$ of the nonlinear Schrödinger (NLS) equation in terms of the solution of an oscillatory RH problem; thus in order to compute the limit of $q(x,t)$ as $t \to \infty$ and $\frac{x}{t} = O(1)$, we must study the large-t asymptotics of the associated RH problem. The corresponding problem for the full line was first studied in [112] (see also [113], [114], and the review [102]). A rigorous and elegant method for studying the asymptotic behavior of oscillatory RH problems was introduced by Deift and Zhou [67], [68]. This method, which can be considered as a nonlinear steepest descent method, can be *immediately* applied to the RH problem of Theorem 16.1. Indeed, after approximating $\Gamma(k)$ by a suitable rational function, it is straightforward to show that the jumps J_1 and J_3 can be absorbed into the jump J_4. This yields an RH problem with a jump $\tilde{J}$ on the real axis, where $\tilde{J}$ is obtained from J_4 by replacing $\gamma(k)$ with $\gamma(k) - \lambda\overline{\Gamma}(k)$. This RH problem is *identical* to the RH problem for the initial-value problem of the NLS, and thus the results of Deift and Zhou are immediately applicable. Similar considerations are valid for other integrable nonlinear PDEs; see [64], [65], [66].

For the NLS the following result is valid; see [64], [50].

Theorem 19.1. Suppose that the conditions of Theorem 16.1 are satisfied. Then the solution $q(x,t)$ of the NLS equation (see (81)) on the half-line corresponding to the initial-boundary values $q_0(x)$, $g_0(t)$, and $g_1(t)$ exhibits the following large-t behavior:

Define the set $\{\kappa_j\}_{j=1}^N$ by $\kappa_j = \lambda_j$, $j = 1,\dots,\Lambda$ and $\kappa_{\Lambda+j} = k_j$, $j = 1,\dots,4$, where λ_j and k_j are defined as in Theorem 16.1.

(i) If the set $\{\lambda_j\}_{j=1}^\Lambda$ is empty, then the asymptotics has a quasi-linear dispersive character, namely it is described by the Zakharov–Manakov-type formulae

$$q(x,t) = t^{-\frac{1}{2}}\alpha\left(-\frac{x}{4t}\right)\exp\left\{\frac{ix^2}{4t} - 2i\lambda\alpha^2\left(-\frac{x}{4t}\right)\log t + i\phi\left(-\frac{x}{4t}\right)\right\} + o\left(t^{-\frac{1}{2}}\right),$$

$$t \to \infty, \quad \frac{x}{4t} = O(1), \tag{19.1}$$

with the amplitude α and the phase ϕ given by the equations

$$\alpha^2(k) = -\frac{\lambda}{4\pi}\log\left(1 - \lambda|\gamma(k) - \lambda\overline{\Gamma}(k)|^2\right), \tag{19.2}$$

$$\begin{aligned}\phi(k) = {} & -6\lambda\alpha^2(k)\log 2 + \frac{\pi(2-\lambda)}{4} + \arg\big(\gamma(k) - \lambda\overline{\Gamma}(k)\big) + \arg G(2i\lambda\alpha^2(k)) \\ & -4\lambda\int_{-\infty}^{k}\log|\mu - k|\,d\alpha^2(\mu),\end{aligned} \tag{19.3}$$

where $G(z)$ denotes Euler's gamma function.

(ii) If $\lambda = -1$ and the set $\{\lambda_j\}_{j=1}^\Lambda$ is not empty, then solitons, which are moving away from the boundary, are generated. This means that there are Λ directions on the (x,t)-plane, namely

$$t \to \infty, \quad -\frac{x}{4t} = \xi_j + O\left(\frac{1}{t}\right), \quad j \in \{1,\dots,\Lambda\}, \tag{19.4}$$

along which the asymptotics is given by the one-soliton formula

$$q(x,t) = -\frac{2\eta_j\exp[-2i\xi_j x - 4i(\xi_j^2 - \eta_j^2)t - i\phi_j]}{\cosh[2\eta_j(x + 4\xi_j t) - \Delta_j]} + O\left(t^{-\frac{1}{2}}\right), \tag{19.5}$$

where

$$\eta_j = \mathrm{Im}\,(\kappa_j), \quad \xi_j = \mathrm{Re}\,(\kappa_j)$$

and the parameters ϕ_j and Δ_j are defined by the following equations:

$$\begin{aligned}\phi_j = {} & -\frac{\pi}{2} + \arg c_j + \sum_{l=1, l\neq j}^{N}\big[1 - \mathrm{sign}(\xi_l - \xi_j)\big]\arg\left(\frac{\lambda_j - \kappa_l}{\lambda_j - \bar{\kappa}_l}\right) \\ & + \frac{1}{\pi}\int_{-\infty}^{-x/4t}\frac{\log\left(1 - \lambda|\gamma(k) - \lambda\overline{\Gamma}(k)|^2\right)}{(\mu - \xi_j)^2 + \eta_j^2}(\mu - \xi_j)\,d\mu,\end{aligned} \tag{19.6}$$

$$\Delta_j = -\log 2\eta_j + \log|c_j| + \sum_{l=1, l\neq j}^{N}\big[1 - \mathrm{sign}(\xi_l - \xi_j)\big]\log\left|\frac{\lambda_j - \kappa_l}{\lambda_j - \bar{\kappa}_l}\right|$$

$$-\frac{\eta_j}{\pi}\int_{-\infty}^{-x/4t}\frac{\log\left(1-\lambda|\gamma(k)-\lambda\overline{\Gamma}(k)|^2\right)}{(\mu-\xi_j)^2+\eta_j^2}d\mu. \tag{19.7}$$

Away from the rays (19.4) the asymptotics again has dispersive character, and it can be described by formulae (19.1)–(19.3), evaluated at $\lambda=-1$, but with the terms

$$\phi_{\text{solitons}}=2\sum_{j=1}^{N}\arg(\kappa_j-k)\big[\text{sign}(\xi_j-k)-1\big]$$

added to the RHS of (19.3).

Proof. Without loss of generality (see [68]) we assume that the functions $\gamma(k)$ and $\Gamma(k)$ can be approximated by rational functions with appropriately chosen poles so that these functions are analytic and bounded in certain domains of the complex k-plane.

We will first show that under the assumptions of Theorem 16.1 and provided that the associated discrete spectrum is empty, the solution $M(x,t,k)$ of the RH problem defined in Theorem 16.1 satisfies

$$M(x,t,k)=\left[I+O\left(\frac{1}{\sqrt{t}}\right)\right][\delta(k)]^{\sigma_3},\quad t\to\infty,$$

$$0<c_1\le\frac{x}{t}\le c_2<\infty, \tag{19.8}$$

uniformly for $|\text{Im}\,k|\ge\varepsilon>0$, where the scalar function $\delta(k)$ is defined by

$$\delta(k)=\exp\left[\frac{1}{2\pi i}\int_{-\infty}^{v}\ln\left[1-\lambda|r(k')|^2\right]\frac{dk'}{k'-k}\right],\quad v=-\frac{x}{4t}, \tag{19.9}$$

with

$$r(k)=\gamma(k)-\lambda\overline{\Gamma(\bar{k})}. \tag{19.10}$$

In order to derive this estimate we must deform the RH problem of Theorem 16.1 to one defined on the steepest descent contours associated with $\exp[2ikx+4ik^2t]$. This deformation was actually carried out in [64]. In what follows we will simplify this deformation by mapping the RH problem of Theorem 16.1 to an RH problem formulated on the real axis: If $\Gamma(k)$ can be approximated by a rational function, then the jumps J_1 and J_3 can be absorbed into the jump J_4, and in this way the RH problem of Theorem 16.1 reduces to the following RH problem:

$$M_-(x,t,k)=M_+(x,t,k)\tilde{J}(x,t,k),\quad k\in\mathbb{R}, \tag{19.11a}$$

$$M=I+O\left(\frac{1}{k}\right),\quad k\to\infty, \tag{19.11b}$$

where M_- and M_- are analytic for $\text{Im}\,k<0$ and $\text{Im}\,k>0$, respectively, and the jump matrix $\tilde{J}$ is defined by

$$\tilde{J}=\begin{pmatrix}1 & -r(k)e^{-\theta}\\ \lambda\bar{r}(k)e^{\theta} & 1-\lambda|r(k)|^2\end{pmatrix},\quad k\in\mathbb{R};\ \ \theta=2i(xk+2k^2t). \tag{19.12}$$

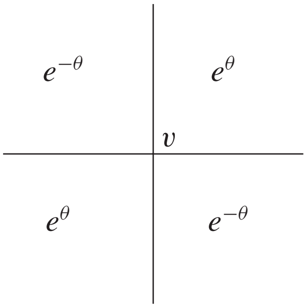


Figure 19.1. *Domains where e^{θ} and $e^{-\theta}$ are bounded.*

The stationary point associated with $\exp[\theta]$ is given by $k = v$, and the directions of the steepest descent are given by $\mathrm{Im}\,[i(k-v)^2] = 0$. Indeed, $\exp[\theta]$ can be rewritten as

$$e^{\theta} = e^{-4itv^2} e^{4it(k-v)^2}.$$

The domains in the complex k-plane where $\exp[\theta]$ and $\exp[-\theta]$ are bounded, are shown in Figure 19.1, while the contours of steepest descent are depicted by the solid lines in Figure 19.2. Deift and Zhou introduced the following construction for deforming the RH problem with a jump across the real line to an RH problem with a jump across the steepest descent contours.

For $v < k < \infty$ we factorize the matrix $\tilde{J}$ in the form

$$\tilde{J} = \begin{pmatrix} 1 & \\ \lambda \bar{r} e^{\theta} & 1 \end{pmatrix} \begin{pmatrix} 1 & -re^{-\theta} \\ 0 & 1 \end{pmatrix}$$

and then we rewrite (19.11a) in the form

$$M_- \begin{pmatrix} 1 & re^{-\theta} \\ 0 & 1 \end{pmatrix} = M_+ \begin{pmatrix} 1 & 0 \\ \lambda \bar{r} e^{\theta} & 1 \end{pmatrix}, \quad v < k < \infty. \tag{19.13}$$

The functions $\exp[-\theta]$ and $\exp[\theta]$ are bounded in the domains 6 and 1 of Figure 19.2, respectively, thus the associated matrices can be absorbed in M_- and M_+, respectively, and therefore the jump across $v < k < \infty$ can be eliminated. However, this strategy fails for $-\infty < k < v$, since $\exp[-\theta]$ and $\exp[\theta]$ are *not* bounded in the domains 4 and 3, respectively. This difficulty can be bypassed by introducing the function $\delta(k)$. Indeed, this function is determined by the requirement that the jump across $-\infty < k < v$ can be eliminated: Let us define $W(x,t,k)$ by

$$W(x,t,k) = M(x,t,k)[\delta(k)]^{-\sigma_3}, \tag{19.14}$$

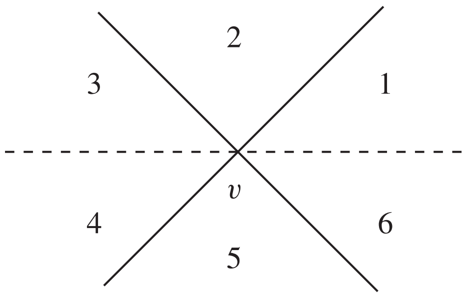


Figure 19.2. *The steepest descent contours for the NLS.*

and let W_-, W_+, δ_-, δ_+ denote the limits of W and δ as k approaches $-\infty < k < v$. Then, (19.11) and (19.14) imply

$$W_+ = W_- \begin{pmatrix} (1-\lambda|r|^2)\frac{\delta_-}{\delta_+} & \delta_+\delta_- r e^{-\theta} \\ -\frac{\lambda\bar{r}}{\delta_+\delta_-}e^{\theta} & \frac{\delta_+}{\delta_-} \end{pmatrix}, \quad -\infty < k < v. \tag{19.15}$$

Thus, if we choose $\delta(k)$ such that

$$\left(1-\lambda|r(k)|^2\right)\frac{\delta_-(k)}{\delta_+(k)} = 1, \quad -\infty < k < v,$$

$$\delta(k) = O\left(\frac{1}{k}\right), \quad k \to \infty, \tag{19.16}$$

(19.15) becomes

$$W_+ = W_- \begin{pmatrix} 1 & \frac{r\delta_+^2}{1-\lambda|r|^2}e^{-\theta} \\ -\frac{\lambda\bar{r}}{(1-\lambda|r|^2)\delta_-^2}e^{\theta} & 1-\lambda|r|^2 \end{pmatrix}, \quad -\infty < k < v.$$

This equation can be rewritten in the form

$$W_+ \begin{pmatrix} 1 & -\frac{r\delta^2}{1-\lambda r\bar{r}}e^{-\theta} \\ 0 & 1 \end{pmatrix} = W_- \begin{pmatrix} 1 & 0 \\ -\frac{\lambda\bar{r}}{(1-\lambda r\bar{r})\delta^2}e^{\theta} & 1 \end{pmatrix}, \quad -\infty < k < v,$$

where $\bar{r}$ denotes $\overline{r(\bar{k})}$. Now the matrices containing $\exp[-\theta]$ and $\exp[e^{\theta}]$ *can* be absorbed in W_+ and W_-, respectively, and here the jump across $-\infty < k < v$ can be eliminated.

The unique solution of the RH problem (19.16) is given by the RHS of (19.9).

In summary, let us define the matrix $X(x,t,k)$ by

$$X(x,t,k) = W(x,t,k)K(x,t,k), \tag{19.17}$$

where W is defined in terms of M by (19.14), and the matrix K has the following form in the domains $1, \dots, 6$:

$$
\begin{aligned}
&1: \begin{pmatrix} 1 & 0 \\ \frac{\lambda \bar{r}}{\delta^2} e^{\theta} & 1 \end{pmatrix} \qquad 2: \begin{pmatrix} 1 & 0 \\ \frac{r}{\delta^2} e^{\theta} & 1 \end{pmatrix} \\
&3: \begin{pmatrix} 1 & -\frac{r\delta^2}{1-\lambda r \bar{r}} e^{-\theta} \\ 0 & 1 \end{pmatrix} \quad 4: \begin{pmatrix} 1 & 0 \\ -\frac{\lambda \bar{r}}{(1-\lambda r\bar{r})\delta^2} e^{\theta} & 1 \end{pmatrix} \\
&5: \begin{pmatrix} 1 & -\lambda \bar{r} \delta^2 e^{-\theta} \\ 0 & 1 \end{pmatrix} \quad 6: \begin{pmatrix} 1 & r\delta^2 e^{-\theta} \\ 0 & 1 \end{pmatrix}.
\end{aligned}
\tag{19.18}
$$

Then X satisfies the following RH problem:

$$
\begin{aligned}
&X_-(x,t,k) = X_+(x,t,k) G(x,t,k), \quad k \in \mathcal{L}, \\
&X = I + O\left(\frac{1}{k}\right), \quad k \to \infty,
\end{aligned}
\tag{19.19}
$$

where $\mathcal{L}$ is the union of the rays $\arg(k - v) = \frac{(2j-1)\pi}{4}$, $j = 1, 2, 3, 4$, and G is defined as follows:

$$
\begin{aligned}
G_1 &= \begin{pmatrix} 1 & 0 \\ -\frac{\lambda \bar{r}}{\delta^2} e^{\theta} & 1 \end{pmatrix}, \qquad \arg(k - v) = \frac{\pi}{4}, \\
G_2 &= \begin{pmatrix} 1 & \frac{r\delta^2}{1-\lambda r\bar{r}} e^{-\theta} \\ 0 & 1 \end{pmatrix}, \qquad \arg(k - v) = \frac{3\pi}{4}, \\
G_3 &= \begin{pmatrix} 1 & 0 \\ \frac{\lambda \bar{r}}{(1-\lambda r\bar{r})\delta^2} e^{\theta} & 1 \end{pmatrix}, \quad \arg(k - v) = \frac{5\pi}{4}, \\
G_4 &= \begin{pmatrix} 1 & -r\delta^2 e^{-\theta} \\ 0 & 1 \end{pmatrix}, \qquad \arg(k - v) = \frac{7\pi}{4}.
\end{aligned}
\tag{19.20}
$$

The RH problem for X has the crucial property that its jump matrices decay exponentially to the identity, provided that k is away from the point v. This implies

$$
X(x,t,k) = I + O\left(\frac{1}{\sqrt{t}}\right), \quad t \to \infty,
\tag{19.21}
$$

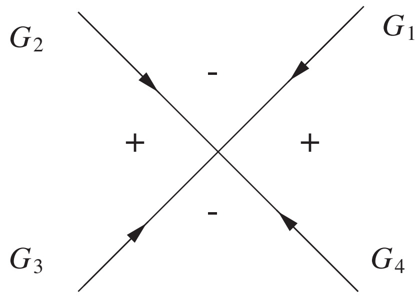


Figure 19.3. *The RH problem for X.*

for k away from v. Indeed,

$$X(x,t,k) = I + \frac{1}{2i\pi}\int_{\mathcal{L}} X_+(x,t,k')(I - G(x,t,k'))\frac{dk'}{k'-k}, \quad k \in \mathbb{C}\backslash\mathcal{L}. \tag{19.22}$$

It turns out that X^+ satisfies the a priori estimate $|X^+(x,t,k)| \leq$ constant; using this estimate and employing the classical Laplace method in the integral on the RHS of the above equation, we find (19.21).

Equations (19.14), (19.17) and the estimate (19.21), together with (16.34), imply that

$$q(x,t) = O\left(\frac{1}{\sqrt{t}}\right), \quad t \to \infty, \quad 0 \leq c_1 \leq \frac{x}{t} \leq c_2 < \infty. \tag{19.23a}$$

In the case that the discrete spectrum is *not* empty, (19.23a) is modified by

$$q(x,t) = q_s(x,t) + O\left(\frac{1}{\sqrt{t}}\right), \tag{19.23b}$$

where q_s represents the soliton contribution [64].

Just as in the classical steepest descent method, one can improve the estimate (19.21) by calculating in closed form the contribution from the stationary point v. It turns out that the corresponding RH problem can be solved in closed form in terms of parabolic cylindrical functions (see [114]). The relevant analysis is very similar to that for the full line problem presented in [102]. This leads to formula (19.1). □

Remark 19.1. The zeros k_j, $j = 1, \ldots, n_1$, of the function $a(k)$ lying in the first quadrant participate in the residue conditions of the RH problem, but they do not generate solitons (there are exactly Λ and not $N = n_1 + \Lambda$ soliton rays indicated in (19.4)). Thus, they affect formulae (19.6) and (19.7) describing the parameters of the soliton (19.5) (the summations in the RHS of these formulae run from 1 to N). A qualitative explanation of the absence in the asymptotics of the solitons corresponding to k_j is quite simple: These solitons move to the left, and hence after a finite time disappear form the first quadrant.

Remark 19.2. In the cases of the linearizable boundary conditions all the parameters in the above formulae can be expressed in terms of the spectral functions $a(k)$ and $b(k)$, i.e., in terms of the initial data only. Indeed, for the cases of $q(0,t) = 0$, $q_x(0,t)$, or $q_x(0,t) - \chi q(0,t) = 0$, we have the following formulae:

$$c_j = \frac{\lambda \overline{b(-\bar{\lambda}_j)}}{a(\lambda_j)\dot{\Delta}_0(\lambda_j)}, \qquad j = 1,\ldots,\Lambda,$$

or

$$c_j = -\frac{\lambda \overline{b(-\bar{\lambda}_j)}}{a(\lambda_j)\dot{\Delta}_1(\lambda_j)}, \qquad j = 1,\ldots,\Lambda,$$

or

$$c_j = -\frac{\lambda \frac{2\lambda_j - i\chi}{2\lambda_j + i\chi}\overline{b(-\bar{\lambda}_j)}}{a(\lambda_j)\dot{\Delta}_\chi(\lambda_j)}, \qquad j = 1,\ldots,\Lambda.$$

Also,

$$\Gamma(k) = \frac{\lambda \overline{b(-\bar{k})}}{a(k)\Delta_0(k)}, \qquad k \in \mathbb{R}^- \cup i\mathbb{R}^+,$$

or

$$\Gamma(k) = -\frac{\lambda \overline{b(-\bar{k})}}{a(k)\Delta_1(k)}, \qquad k \in \mathbb{R}^- \cup i\mathbb{R}^+,$$

or

$$\Gamma(k) = -\frac{\lambda \frac{2k - i\chi}{2k + i\chi}\overline{b(-\bar{k})}}{a(k)\Delta_\chi(k)}, \qquad k \in \mathbb{R}^- \cup i\mathbb{R}^+.$$

19.2 Asymptotics in Transient Stimulated Raman Scattering

Under the physical assumptions described in [61], the phenomenon of transient stimulated Raman scattering can be described by the solution of a certain initial-boundary value problem for a system of three nonlinear PDEs for the complex-valued functions $X(\chi,\tau)$, $Y(\chi,\tau)$ and for the real-valued function $b(\chi,\tau)$, where the independent variables χ and τ are real. These equations are the compatibility condition of the following Lax pair:

$$\psi_\chi(\chi,\tau,k) = \begin{pmatrix} -ik & X(\chi,\tau) \\ -\bar{X}(\chi,\tau) & ik \end{pmatrix} \Psi(\chi,\tau,k) \tag{19.24a}$$

and

$$\psi_\tau(\chi,\tau,k) = \frac{1}{4k}\begin{pmatrix} ib(\chi,\tau) & -Y(\chi,\tau) \\ \bar{Y}(\chi,\tau) & -ib \end{pmatrix} \Psi(\chi,\tau,k), \tag{19.24b}$$

where $k \in \mathbb{C}$. The solution of the relevant initial-boundary value problem can be expressed through the solution of the matrix RH problem described below in Theorem 19.2. This problem can be obtained by following the steps 1 and 2 described in Chapter 16. Actually, in contrast to the case of NLS, sine-Gordon (sG), Korteweg–de Vries (KdV), and modified KdV (mKdV), where the associated RH problem depends on some *unknown* boundary values, the RH problem of Theorem 19.2 involves only the *known* initial and boundary data. Hence, this RH problem provides directly the *effective* solution *without* the need to analyze the associated global relation.

For brevity of presentation, in what follows we state, rather than derive, the relevant RH problem. The derivation can be found in [61]; rigorous aspects are discussed in [63].

Theorem 19.2. Let $b(\chi,\tau)\in\mathbb{R}$, $Y(\chi,\tau)\in\mathbb{C}$, and $X(\chi,\tau)\in\mathbb{C}$ satisfy the following equations with $\chi\in[0,l]$, $l>0$, and $\tau\in[0,1]$:

$$\frac{\partial b}{\partial \chi}=i(\bar{X}Y-X\bar{Y}),\qquad \frac{\partial Y}{\partial \chi}=2ibX,\qquad \frac{\partial X}{\partial \tau}=-\frac{i}{2}Y. \tag{19.25}$$

Let

$$b(0,\tau)=b_0(\tau),\quad Y(0,\tau)=Y_0(\tau),\quad X(\chi,0)=X_0(\chi), \tag{19.26a}$$

where $b_0(\tau)$, $Y_0(\tau)$ are differentiable for $\tau\in[0,1]$, and $X_0(\chi)$ is differentiable for $\chi\in[0,l]$. Assume that

$$b_0(\tau)^2+|Y_0(\tau)|^2=1. \tag{19.26b}$$

The unique solution of this initial-boundary value problem is given by

$$X(\chi,\tau)=2i\lim_{k\to\infty}\left[kM_{11}(\chi,\tau,k)\right],$$

$$b(\chi,\tau)=-1-4i\frac{\partial}{\partial\tau}\lim_{k\to\infty}\left[\overline{kM_{21}(\chi,\tau,\bar{k})}\right],\quad k\in\mathbb{C},\quad k_1\neq 0, \tag{19.27}$$

where the scalar functions $M_{11}(\chi,\tau,k)$ and $M_{21}(\chi,\tau,k)$, $k\in\mathbb{C}$, can be obtained by solving the following RH problem:

$$M_+(\chi,\tau,k)=M_-(\chi,\tau,k)\begin{pmatrix}1 & \frac{\rho_2(k)}{\rho_1(k)}e^{2ik\chi+\frac{i\tau}{2k}}\\ -\frac{\overline{\rho_2(k)}}{\overline{\rho_1(k)}}e^{-2ik\chi-\frac{i\tau}{2k}} & \frac{1}{|\rho(k)|^2}\end{pmatrix},\quad k\in\mathbb{R}, \tag{19.28a}$$

$$M(x,t,k)=I+O\left(\frac{1}{k}\right),\quad k\to\infty,\quad k_I\neq 0. \tag{19.28b}$$

This RH problem, which is specified through the scalar functions $\rho_1(k)$ and $\rho_2(k)$, $k\in\mathbb{R}$, has a unique solution. The functions $\rho_1(k)$ and $\rho_2(k)$ can be constructed in terms of the given initial and boundary data as follows: Let $(\mu_1(\tau,k),\mu_2(\tau,k))$ be the unique solution of

$$\frac{\partial}{\partial\tau}\begin{pmatrix}\mu_1(\tau,k)\\ \mu_2(\tau,k)\end{pmatrix}=\frac{1}{4k}\begin{pmatrix}ib_0(\tau) & -Y_0(\tau)\\ \bar{Y}_0(\tau) & -ib_0(\tau)\end{pmatrix}\begin{pmatrix}\mu_1(\tau,k)\\ \mu_2(\tau,k)\end{pmatrix}, \tag{19.29a}$$

$$\mu_1(1,k)=1,\quad \mu_2(1,k)=0. \tag{19.29b}$$

Let $(\nu_1(\chi,k), \nu_2(\chi,k))$ be the unique solution of

$$\frac{\partial}{\partial\chi}\begin{pmatrix}\nu_1(\chi,k)\\ \nu_2(\chi,k)\end{pmatrix}=\begin{pmatrix}-ik & X_0(\chi)\\ -\bar{X}_0(\chi) & ik\end{pmatrix}\begin{pmatrix}\nu_1(\chi,k)\\ \nu_2(\chi,k)\end{pmatrix}, \tag{19.30a}$$

$$\nu_1(0,k)=\mu_1(0,k)e^{-\frac{i}{4k}}, \quad \nu_2(0,k)=\mu_2(0,k)e^{-\frac{i}{4k}}. \tag{19.30b}$$

The functions $\rho_1(k)$ and $\rho_2(k)$ are defined by

$$\rho_1(k)=\nu_1(l,k)e^{ikl}, \quad \rho_2(k)=\nu_2(l,k)e^{-ikl}. \tag{19.31}$$

Proof. The derivation can be found in [61]. □

In order to study the large-χ asymptotics of the above initial-boundary value problem it is more convenient to rewrite the RH problem (19.28) in terms of a system of linear integral equations. Indeed, if $\rho_1(k)\neq 0$ for $\operatorname{Im} k \geq 0$, the above RH problem reduces to solving a system of linear integral equations. In this case, $X(\chi,\tau)$ and $b(\chi,\tau)$ are given by

$$X(\chi,\tau)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\bar{\rho}_2(k)}{\bar{\rho}_1(k)}e^{-2ik\chi-\frac{i\tau}{2k}}M_1(\chi,\tau,k)dk,$$

$$b(\chi,\tau)=-1+\frac{2}{\pi}\frac{\partial}{\partial\tau}\int_{-\infty}^{\infty}\frac{\rho_2(k)}{\rho_1(k)}e^{2ik\chi+\frac{i\tau}{2k}}\bar{M}_2(\chi,\tau,k)dk, \tag{19.32}$$

where the functions M_1 and M_2 are defined as the unique solution of the following system of linear integral equations:

$$\begin{pmatrix}-M_2(\chi,\tau,k)\\ M_1(\chi,\tau,k)\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}+\frac{1}{2i\pi}\int_{-\infty}^{\infty}\frac{\rho_2(k')}{\rho_1(k')}e^{2ik'\chi+\frac{i\tau}{2k'}}\begin{pmatrix}\bar{M}_1(\chi,\tau,k')\\ \bar{M}_2(\chi,\tau,k')\end{pmatrix}\frac{dk'}{k'-(k-i0)}. \tag{19.33}$$

If $\rho_1(k_j)=0$, $j=1,2,\ldots$, $\operatorname{Im} k_j \geq 0$, then the above RH problem reduces to solving a system of linear integral equations similar to (19.33) supplemented with a system of algebraic equations.

If we attempt to solve the linear integral equations (19.33) by iteration, we find the integral

$$J(\chi,\tau)=\int_{-\infty}^{\infty}e^{2ik\chi+\frac{i\tau}{2k}}f(k)dk. \tag{19.34}$$

The asymptotic evaluation of such integrals is well established; see, for example, [17]. There exist two important cases:

(a) If $\chi\to\infty$ and $\tau/\chi=O(1)$, the stationary phase method implies that $J\sim O(\frac{1}{\sqrt{\chi}})$. This is precisely the limit analyzed in section 19.1. In that case, if solitons are absent, the associated RH problem (which is equivalent to a system of linear integral equations similar to (19.33)) implies that $q(x,t)$ is indeed of $O(\frac{1}{\sqrt{t}})$ (see (19.23a)); if solitons are present, then the asymptotic behavior of the solution is dominated by solitons; see (19.23b).

(b) If $\chi \to \infty$ and $\tau/\chi = o(1)$, then there exists a moving stationary point, and one introduces the *similarity variables* $\xi = \sqrt{\tau\chi}$, $k = \frac{1}{2}\sqrt{\frac{\tau}{\chi}}\lambda$. Then

$$J = \frac{1}{2}\sqrt{\frac{\tau}{\chi}}\int_{-\infty}^{\infty} e^{i\xi(\lambda+\frac{1}{\lambda})} f\left(\frac{1}{2}\sqrt{\frac{\tau}{\chi}}\lambda\right) d\lambda \tag{19.35}$$

and the leading behavior of the integral depends on the limit of $f(k)$ as $k \to 0$. This case is also relevant in the usual soliton systems, but it characterizes only a certain transition zone. It is important to note that in our case $\tau \in [0,1]$ and $\chi \to \infty$; thus, $\tau/\chi = o(1)$ and the asymptotic behavior of the system is dominated by the underlying similarity solution. Actually, in our case even if solitons are present, i.e., even if there exist points $k_1, k_2, \ldots$ in $\mathbb{C}^+$ such that $\rho_1(k_j) = 0$, the large-χ asymptotics is still dominated by the similarity solution because the contribution from the solitons is exponentially small.

Theorem 19.3. Consider the initial-boundary value problem defined in Theorem 19.2 but with $X_0(\chi) = 0$. The leading order behavior as $\chi \to \infty$ of the solution of this problem is given by

$$X(\chi,\tau) = \frac{1}{2}\frac{\tau}{\xi}\tilde{X}(\xi), \quad b(\chi,\tau) = -1 + \frac{1}{4}\frac{\tilde{b}(\xi)}{\xi} + \frac{1}{4}\frac{d}{d\xi}\tilde{b}(\xi), \quad \xi = \sqrt{\tau\chi}, \tag{19.36}$$

where

$$\tilde{X}(\xi) = -\frac{i}{\pi}\frac{Y_0(0)}{1-b_0(0)}\int_{-\infty}^{\infty} e^{-i\xi(\lambda+\frac{1}{\lambda})} N_1(\xi,\lambda)d\lambda,$$

$$\tilde{b}(\xi) = \frac{2i}{\pi}\frac{\bar{Y}_0(0)}{1-b_0(0)}\int_{-\infty}^{\infty} e^{i\xi(\lambda+\frac{1}{\lambda})} \bar{N}_2(\xi,\lambda)d\lambda, \tag{19.37}$$

and the functions N_1, N_2 are the unique solution of the system of linear integral equations

$$\begin{pmatrix} -N_2(\xi,\lambda) \\ N_1(\xi,\lambda) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$+\frac{1}{2\pi}\frac{\bar{Y}_0(0)}{1-b_0(0)}\int_{-\infty}^{\infty} e^{i\xi(\lambda'+\frac{1}{\lambda'})} \begin{pmatrix} \bar{N}_1(\xi,\lambda') \\ \bar{N}_2(\xi,\lambda') \end{pmatrix} \frac{d\lambda'}{\lambda' - (\lambda - i0)}. \tag{19.38}$$

This system is a particular case of a more general system of linear integral equations which characterizes the general solution of the Painlevé III equation in [115].

Proof. Let

$$k = \frac{1}{2}\sqrt{\frac{\tau}{\chi}}\lambda, \quad \xi = \sqrt{\tau\chi}.$$

Then the kernel of (19.33) becomes

$$\frac{\rho_2\left(\frac{1}{2}\sqrt{\frac{\tau}{\chi}}\lambda'\right)}{\rho_1\left(\frac{1}{2}\sqrt{\frac{\tau}{\chi}}\lambda'\right)} e^{i\xi(\lambda'+\frac{1}{\lambda'})} \frac{d\lambda'}{\lambda' - (\lambda - i0)}.$$

Thus the leading order behavior of (19.33) as $\chi \to \infty$ depends on the limit of $\rho_2(k)/\rho_1(k)$ as $k \to 0$. The analysis of (19.30) and the definitions of ρ_1, ρ_2 (see (19.31)) imply (see [61] for details)

$$\frac{\rho_2(k)}{\rho_1(k)} \sim \frac{\alpha_2 + \beta_2 e^{-\frac{i}{2k}}}{\alpha_1 + \beta_1 e^{-\frac{i}{2k}}}, \quad k \to 0,$$

where $\alpha_1, \beta_1, \alpha_2, \beta_2$ are certain constants. It is interesting that the terms involving $\exp(-i/2k)$ give no contribution. Indeed,

$$\frac{\rho_2(k)}{\rho_1(k)} e^{2ik\chi + \frac{i\tau}{2k}} \sim \frac{\alpha_2}{\alpha_1}\left[e^{2ik\chi+\frac{i\tau}{2k}} + \frac{\left(\frac{\beta_2}{\alpha_2} - \frac{\beta_1}{\alpha_1}\right) e^{2ik\chi - \frac{i}{2k}(1-\tau)}}{1 + \frac{\beta_1}{\alpha_1} e^{-\frac{i}{2k}}}\right], \quad k \to 0.$$

If $\alpha_1 + \beta_1 e^{-\frac{i}{2k}} \neq 0$ for $\operatorname{Im} k \geq 0$, then because of analyticity in $\mathbb{C}^+$ (since $\tau \leq 1$, the exponential terms decay in $\mathbb{C}^+$), these terms give zero contribution to (19.33); $\bar{M}_1$, $\bar{M}_2$, and $[k' - (k - i0)]^{-1}$ are also analytic in $\mathbb{C}^+$. If $\alpha_1 + \beta_1 e^{-\frac{i}{2k}} = 0$, the extra terms due to the poles give a contribution that is exponentially small as $\chi \to \infty$.

The above analysis implies that the leading behavior of (19.33) as $\chi \to \infty$ is characterized by

$$\begin{pmatrix} -N_2(\xi,\lambda) \\ N_1(\xi,\lambda) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{2i\pi}\frac{\alpha_2}{\alpha_1}\int_{-\infty}^{\infty} e^{i\xi(\lambda'+\frac{1}{\lambda'})} \begin{pmatrix} \bar{N}_1(\xi,\lambda') \\ \bar{N}_2(\xi,\lambda') \end{pmatrix} \frac{d\lambda'}{\lambda' - (\lambda - i0)}. \tag{19.39}$$

We emphasize that even if $\rho(k)$ has zeros for $\operatorname{Im} k > 0$, these zeros do not contribute to the leading behavior of the solution of the RH problem (19.28). Indeed, if there exist zeros, (19.33) has to be supplemented with certain additional terms. However, these terms vanish exponentially as $\chi \to \infty$. Consider for simplicity the case of one zero, $\rho(k_1) = 0$; the extension to any number of zeros is straightforward. If $\rho(k_1) = 0$, $\operatorname{Im} k_1 > 0$, the RHS of (19.33) also contains the term

$$-\frac{\rho_2(k_1) e^{2ik_1\chi + \frac{i\tau}{2k_1}}}{\dot{\rho}(k_1)(k - k_1)} \begin{pmatrix} \bar{M}_1(\chi,\tau,k_1) \\ \bar{M}_2(\chi,\tau,k_2) \end{pmatrix}, \quad \dot{\rho}(k_1) = \left.\frac{d\rho}{dk}\right|_{k=k_1},$$

where $(\bar{M}_1, \bar{M}_2)$ is given in terms of a certain linear integral equation whose kernel involves $(k' - k_1)^{-1}$. Since the term $e^{2ik_1\chi}$ with $\operatorname{Im} k_1 > 0$ is exponentially small, it follows that the zeros give an exponentially small contribution as $\chi \to \infty$.

The above analysis is valid even if $X(\chi, 0) \neq 0$. In the particular case that $X(\chi, 0) = 0$, it is shown in [61] that the constants α_1 and α_2 satisfy

$$\alpha_2 = \frac{i\bar{Y}(0,0)}{1 - b(0,0)}\alpha_1. \tag{19.40}$$

Equations (19.39) and (19.40) yield (19.38). Equations (19.32) imply

$$X = \frac{1}{2}\sqrt{\frac{\tau}{\chi}}\tilde{X}(\xi), \quad b = -1 + \frac{\partial}{\partial \tau}\left(\frac{1}{2}\sqrt{\frac{\tau}{\chi}}\tilde{b}(\xi)\right), \tag{19.41}$$

where $\tilde{X}(\xi)$ and $\tilde{b}(\xi)$ are defined by (19.37) (since $\rho_2/\rho_1 \sim \alpha_2/\alpha_1 = i\bar{Y}/(1-b)$). Using

$$\sqrt{\frac{\tau}{\chi}} = \frac{\tau}{\xi}, \quad \frac{\partial}{\partial \tau} = \frac{1}{2}\sqrt{\frac{\chi}{\tau}}\frac{\partial}{\partial \xi},$$

(19.41) become (19.36). □

Epilogue

A unified approach for analyzing boundary value problems for linear and for integrable nonlinear PDEs in two dimensions has been presented.

Linear PDEs

The main advantage of the new method is that it constructs *integral* representations for the solution of a large class of initial-boundary value problems. It is surprising that the method yields novel representations even for the classical boundary value problems for the heat, Laplace, and Helmholtz equations, which are traditionally solved through *series* representations. The novel *integral* representations presented in this book are formulated in the complex k-plane. Hence, through suitable contour deformations, these solutions can be expressed through integrals involving integrands which decay exponentially. This has both analytical and computational advantages.

For the classical elliptic PDEs formulated in the interior of an equilateral or an isosceles orthogonal triangle the new method, in addition to constructing integral as opposed to series [116] representations, also yields explicit solutions for certain oblique Robin boundary conditions (see [23], [24]), for which explicit solutions were not known until now.

The emphasis of the new method has been on the construction of analytic formulae. However, the method also has implications for the numerical evaluation of the solution of initial-boundary value problems for evolution PDEs as well as for the numerical evaluation of the unknown boundary values of elliptic PDEs formulated in the interior of a convex polygon. Although these numerical results appear to be already competitive, they should be considered exploratory as opposed to definitive. It is expected that further improvements will be made, particularly if these techniques are pursued by other researchers; see, for example, [117].

The goal of this book is to introduce the basic elements of the new method as opposed to presenting an exhaustive list of problems that have been analyzed using this approach. However, we list some of these problems below.

(a) A variety of boundary value problems for the Laplace, biharmonic, modified Helmholtz, and Helmholtz equations are solved in [23], [24], [25], [29], [30], [31], [32], [33], [91].

(b) Several boundary value problems for the Laplace and Helmholtz equations in the interior of a wedge and of a circular sector are solved in [118]. The solutions are expressed in terms of *integrals*, in contrast to the *series* representations obtained by the classical methods.

(c) It appears that the new method is also useful for problems involving *linear* PDEs with *nonlinear* boundary conditions. For example, a novel formulation of the classical problem of water waves in $\{0 < x_j < \infty, j = 1, 2, 0 < y < \eta(x_1, x_2, t), t > 0\}$ is presented in [119], and this formulation is used for the derivation of several asymptotic approximations (including the derivation of the Korteweg–de Vries (KdV), nonlinear Schrödinger (NLS), and Kadomtsev–Petviashvili equations), as well as for the numerical computation of lumps for sufficiently large surface tension.

The implementation of the new method to a variety of problems with the emphasis on elliptic boundary value problems will be presented in the forthcoming book of Crowdy and the author [120].

The main limitation of the new method is that it can be applied *only* to those PDEs for which the formal adjoint admits an explicit one-parameter *family of solutions*. This includes PDEs with constant coefficients and separable PDEs with variable coefficients such as (9.41), (9.46), and (9.47).

Several fundamental problems remain open, including the following:

(a) The investigation of possible singularities of the solutions of elliptic PDEs at the corners of a convex polygon.

(b) The study of elliptic PDEs in a nonpolygonal convex domain (this should be achieved by employing the methodology of Chapter 8).

(c) The analytical and numerical study of elliptic PDEs in the *exterior* of a convex polygon. Regarding this problem, the relation of the method presented here with the important works of [121], [122] should be investigated (some preliminary results for the Laplace equation in the exterior of an equilateral triangle are presented in [124]).

Nonlinear PDEs

There exist certain distinctive nonlinear equations called *integrable* (see [125], [126], [127]). The theory of integrable systems has had a significant impact on both mathematics and mathematical physics. Examples include the development of rigorous techniques for the asymptotic evaluation of Riemann–Hilbert (RH) problems [67], the solution of the Schottky problem [128] (using techniques developed by Krichever [129]), the solution of Ulam's problem [130], applications to geometry [131], [132], [133], the rigorous evaluation of the asymptotic behavior of orthogonal polynomials [16], [134], [135], and applications to field theories [136], [137], [138].

Although there exist several types of integrable systems, including ODEs (such as the Painlevé transcendents [139] and several Hamiltonian systems [140]), singular integrodifferential equations (such as the Benjamin–Ono equation [141]), difference equations [142], functional equations [143], and cellular automata [144], *evolution* PDEs have played a crucial role in the development of the theory of integrable systems. Indeed, the modern history of integrable systems begins with the celebrated works of Kruskal and Zabusky [145] on the Cauchy problem of the KdV equation using what was eventually called the *inverse scattering transform method*. The next fundamental step was taken by Lax [146] who understood that the formulation of a *nonlinear* equation as *the compatibility condition of two linear eigenvalue equations*, the so-called Lax pair of equations, is the defining property of an integrable equation. Explosive activity in this new area began only after the crucial work of Zakharov and Shabat [48], who solved the Cauchy problem for the NLS equation (thus establishing that the integrability of the KdV equation was *not* a singular event), and also introduced the powerful RH formalism for the implementation of the inverse scattering method. Soon thereafter the modified KdV [147] and the sG [148] equations were integrated, and the inverse scattering method was declared as a nonlinear analogue of the Fourier transform method [149].

A new method for analyzing initial-boundary value problems was announced in [1], where it was emphasized that this method is based on the *simultaneous* spectral analysis of both parts of the associated Lax pair. This is to be contrasted with the usual inverse scattering transform method which is based on the spectral analysis of *one* of the two equations forming the Lax pair. Actually, this new method is the result of a series of developments: It was first realized by the author [150] that for the solution of initial-boundary value problems, in addition to analyzing the t-independent part of the Lax pair, it is also necessary to analyze the t-part of the Lax pair; this yields $q(x,t)$ in terms of *two* RH problems. These two problems were combined into *one* basic RH problem in the works of the author and Its [64], [65], [66]. These basic RH problems for the NLS and the sG equations are identical to the ones presented in Chapter 16. This made it possible, using the Deift–Zhou method, to obtain long-time asymptotic results. However, the derivation of the basic RH problem was based on the *separate* spectral analysis of the x- and t-parts of the Lax pair and was quite complicated. This derivation was greatly simplified in [1] using the *simultaneous* spectral analysis of the Lax pair. This derivation was further simplified in [151], where it was realized that the best way to implement the simultaneous spectral analysis is to use the formulation of the exact 1-form presented in Chapter 16. Furthermore, this formulation yields the *global relation* in a straightforward manner. The proof that the global relation is not only a necessary but also a sufficient condition for existence, as well as the rigorous investigation of the global relation, was presented in [50].

The most complicated problem in the implementation of the new method is the characterization of the unknown boundary values. The first attempt to characterize the spectral functions involving the unknown boundary values was made in [152] and led to a formal *nonlinear* RH problem. A similar formulation was presented in [153] (apparently the authors of [153] were unaware of [152]). In [153], a different formulation was also presented which was based on an attempt to express the unknown boundary value $q_x(0,t)$ for the NLS equation in terms of the eigenfunction $\Phi(t,k)$ using certain analyticity arguments. However, the formal attempts of [152] and [153] yield a system of *nonlinear Fredholm* integral equations for the spectral functions. The insurmountable problem is that, since the spectral functions are characterized to within an *equivalent class* (recall that $c^+(k)$ in (18.4) is *arbitrary*), one *cannot* rigorously establish solvability for the associated Fredholm equations. An important development in the analysis of the global relation was announced in [57], where it was shown that $q_x(0,t)$ can be characterized in terms of a system of nonlinear *Volterra* as opposed to Fredholm integral equations. The relevant methodology was further simplified and applied to other PDEs in [58]; see Chapter 18.

Several fundamental problems remain open, including the following:

(a) The analytical results obtained in [61] for the model of transient stimulated Raman scattering were compared in [61] with the relevant experimental results (see also [154]). Initial-boundary value problems for integrable PDEs appear in a variety of physical circumstances (see, for example, [155], [156], [157]); a comparison of the experimental with the analytical results remains open.

(b) A particular exact reduction of the Einstein equations, called the Ernst equation, is integrable. The hyperbolic version of this equation belongs to the simpler class of integrable PDEs for which all the boundary values appearing in the spectral functions of the associated RH problem are prescribed as boundary conditions. The formulation of an RH problem for the Ernst equation, which was presented in [158], played an important role in the development of the new method since this was the first time that the *simultaneous* spectral analysis of both parts of the Lax pair was performed. The question of using the asymptotic machinery of RH problems to characterize a certain singularity appearing in Ernst equation remains open.

(c) The investigation of elliptic PDEs, such as the elliptic sG equation [159], using the new method remains open. Preliminary investigations indicate that the new method provides a simpler approach for obtaining the remarkable results of [160] for the elliptic version of the Ernst equation.

(d) Initial-boundary value problems formulated on the finite interval are analyzed in [54], [55]. It turns out that the particular case of *periodic-in-x* boundary conditions yields *linearizable* boundary value problems. The investigation of this particular case and the connection with the classical works of [161], [162], [163] (see also [164]) remain open.

An interesting approach to initial-boundary value problems using some elements of the new method is presented in [165].

Further Developments

In spite of the substantial progress in the analysis of the global relation, the characterization of the unknown boundary values still involves the solution of a *nonlinear* problem. Recently, it has become clear that the most efficient approach for analyzing initial-boundary value problems involves the combination of the following three different formalisms.

(a) Use the PDE techniques pioneered in [166] (see also [99], [100], [101]) to establish the well-posedness of a given initial-boundary value problem.

(b) Use the formalism reviewed in this book to express the solution in terms of an RH problem.

(c) Use the asymptotic techniques of [67], [68], [69] to study the asymptotic properties of the solution.

This approach has the advantage that it uses the essential achievement of the integrable machinery, namely the determination of the asymptotic properties of the solution (asymptotic results *cannot* be obtained so far by standard PDE techniques).

It is shown in [167] that this methodology, in addition to being effective for the case of decaying boundary conditions, can also be used for the physically more significant case of t-periodic boundary conditions.

Multidimensions

It is well known that there *do* exist integrable nonlinear evolution PDEs in *two* spatial dimensions. For instance, physically significant generalizations of the KdV and NLS equations are the Kadomtsev–Petviashvili and Davey–Stewartson equations. A formal method for the solution of the Cauchy problem of these equations was introduced by Ablowitz, the author, and their collaborators using either a nonlocal RH or a d-bar problem [168], [169]; for a rigorous treatment see [170], [171], [172]. The generalization of the method reviewed in this book to three dimensions is in progress. In this respect we note that the following results have already been obtained.

(a) A linear evolution PDE with derivatives of arbitrary order for the scalar function $q(x_1, x_2, t)$ with $\{0 < x_j < \infty,\ j = 1, 2, t > 0\}$ is solved in [2].

(b) The Laplace and Helmholtz equations in a spherical cone are analyzed in [173].

(c) The heat equation with the space variables x_1 and x_2 in the interior of an equilateral triangle is solved in [23].

(d) The Davey–Stewartson equation in the upper half-plane is analyzed in [175].

In conclusion, we note that the interplay of methods for solving linear and integrable nonlinear PDEs appears quite useful [176], [177].

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